18.06, Fall 2004, Problem Set 10

Due before 4PM on **Thursday** December 2nd, 2004, in the boxes in 2-106. No late homework will be accepted. Don't forget to write your name, recitation section **and the names of students you have collaborated with** on the problem set. There is one box for each recitation section. For full credit, please be sure to show and explain your work. Exercises refer to the 3rd edition of the textbook.

Reading assignment: Sections 6.3, 6.4

1. Consider the differential equation $\frac{du}{dt} = Au$ where u is 2-dimensional and
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$
   (a) What are the eigenvalues and eigenvectors of A?
   (b) Compute e^{At}. (Even if the eigenvalues are complex, e^{At} should be real and there should not be any imaginary part in your final expression.)
   (c) Let $u_1(0)$ and $u_2(0)$ be the first and last digit of your MIT ID, respectively. Give an expression for $u_i(t)$ as a function of t.
   (d) Is the differential equation stable?
   (e) In the u_1, u_2 plane, what figure corresponds to the trajectory $(u_1(t), u_2(t))$?

2. Let
$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
   (a) What are the eigenvalues of A?
   (b) How many linearly independent eigenvectors does A have?
   (c) Compute e^{tA}.

3. Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$
   (a) Compute an LDL^T factorization of A where L is lower triangular and D is diagonal.
   (b) What are the eigenvalues and eigenvectors of A? (One of the eigenvalues should be easy to guess as this is a magic square.)
   (c) Give a $Q\Lambda Q^T$ factorization of A, where Q is orthogonal and Λ is diagonal.
   (d) Compute the product of the eigenvalues and the product of the diagonal elements of D. Why are you getting the same answer?

4. Let A be any 3×3 symmetric matrix. Is it true that for large enough t, $A + tI$ is positive definite? Justify.