Questions from 18.06 Final, Fall 2003

1. Suppose $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}, U = \begin{bmatrix} 5 & 0 & 5 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) What are the dimensions of the 4 fundamental subspaces associated with A?

   (b) Give a basis for each of the 4 fundamental subspaces.

$N(A)$

$R(A)$

$C(A)$

$N(A^T)$

2. Let F be the subspace of R^4 given by

$$F = \{(x, y, z, w) : x - y + 2z + 3w = 0\}.$$

Let P be the projection matrix for projecting onto F. (Many of the subquestions can be answered independently of the others.)

(a) Give an orthonormal basis $\{v_1, \cdots, v_k\}$ for the orthogonal complement to F.

(b) Find an orthonormal basis $\{w_1, \cdots, w_l\}$ for F. Explain how you proceed.

The following questions refer to the projection matrix P for projecting onto F.

(c) What are the eigenvalues of P? Give them with their multiplicities.

(d) What is the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto F?

3. (a) Write down the 2×2 rotation matrix, $R(\theta)$, that rotates R^2 in the counterclockwise direction by an angle θ (this matrix is a function of θ).

(b) Compute the eigenvalues of $R(\theta)$. For which value(s) of θ are the eigenvalues real?

(c) What are the eigenvectors of $R(\theta)$.

(d) Write down the singular value decomposition of $R(\theta)$.

4. (a) Give two 3×3 matrices A and B such that AB is not equal to BA.

(b) Suppose A and B are $n \times n$ matrices with the same set of linearly independent eigenvectors $v_1, v_2, \cdots, v_n$. However, the eigenvalues might be different: v_i is the eigenvector for the eigenvalue λ_i of A and the eigenvector for the eigenvalue μ_i of B. Show that $AB = BA$.

5. Consider the differential equation $\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.

(a) Solve the differential equation and express $u(t), v(t)$ as functions of $u(0)$ and $v(0)$.

(b) Find a linear transformation $\begin{bmatrix} p \\ q \end{bmatrix} = T \begin{bmatrix} u \\ v \end{bmatrix}$ such that the differential equation simplifies into two independent differential equations in p and in q (one relating $\frac{dp}{dt}$ and p, the other relating $\frac{dq}{dt}$ and q)

(c) Are there initial conditions $u(0)$, $v(0)$ that would make $u(t)$ blow up? If yes, give one such value for $u(0)$ and $v(0)$.

(d) Are there initial conditions $u(0)$, $v(0)$ that would make $u(t)$ go to 0? If yes, give one such value for $u(0)$ and $u'(0)$.

6. True of False. **Circle** the appropriate answer. "True" means "always true", and "false" means "sometimes false". Justify each answer briefly.

   (a) The product of the pivots when performing Gauss-Jordan is equal to the determinant if we do not have to permute rows.
   True or False.

   (b) Let A be an $m \times n$ matrix whose columns are independent. Then AA^T is positive definite.
   True or False.

   (c) If A is a symmetric matrix then the singular values are the absolute values of the nonzero eigenvalues.
   True or False.

   (d) There exists a 5×5 unitary matrix with eigenvalues 1, $1+i$, $1-i$, i and $-i$.
   True or False.

   (e) Suppose V and W are two vector spaces of dimension n. If T is a linear transformation from V to W with only the 0 vector in the kernel, then for any basis of V, there exists an orthonormal basis of W such that the resulting matrix representing T is upper triangular.
   True or False.

7. Consider the following matrix A:

$$A = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{bmatrix}.$$

(c) Can you immediately tell one of the eigenvalues of A (without computing them)? Explain.

(d) Compute the determinant of A.

(e) Find the eigenvalues of A and the corresponding eigenvectors. (Check your answer.)

(f) Two out of the 3 eigenvectors of A should be orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. How could you have explained this before computing the eigenvectors?

(g) Write an exact expression for A^{100}.

8. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

For each of the following matrices, either complete it (find values for the non-diagonal elements) so that it becomes similar to A, or explain why it is impossible to complete it to a matrix similar to A. Circle whether you are able to complete it or not to a matrix similar to A.

(a)

$$B = \begin{bmatrix} 2 & . & . \\ . & 2 & . \\ . & . & 4 \end{bmatrix}$$

Able to complete it to similar?: Yes No

If yes, give a completion. If not, why not?

(b)

$$C = \begin{bmatrix} 3 & . & . \\ . & 3 & . \\ . & . & 3 \end{bmatrix}.$$

Able to complete it to similar?: Yes No

If yes, give a completion. If not, why not?