Course 18.06, Fall 2002: Quiz 2, Solutions

1 (a) The columns of A are linearly dependent, but the column space is spanned by $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Use this matrix in the formula for the projection matrix:

$$P_C = A(A^TA)^{-1}A^T = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 3 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

(b) As before but with $A = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$:

$$P_R = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

The vector in the row space of A closest to $\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is

$$P_R\boldsymbol{b} = \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(c)

$$P_CA = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = A$$

$$P_CAP_R = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = A$$

The two multiplications project the columns/rows of A onto the column/row space of A. This does not change the matrix.

(d) All vectors orthogonal to the row space of A are in the null space of A. A basis for the nullspace of A is (note dimension 2):

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

2 (a) One choice for the last column is

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

and the normalization constant is

$$c = \frac{1}{2}$$

(b) The projection of $\boldsymbol{b} = (1,1,1,1)$ onto $\boldsymbol{q}_1$ is

$$\boldsymbol{q}_1\boldsymbol{q}_1^T\boldsymbol{b} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The projection of $\boldsymbol{b} = (1,1,1,1)$ onto the plane spanned by $\boldsymbol{q}_1, \boldsymbol{q}_2$ is

$$\boldsymbol{q}_1\boldsymbol{q}_1^T\boldsymbol{b} + \boldsymbol{q}_2\boldsymbol{q}_2^T\boldsymbol{b} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

(c) The first three outputs will be the first three columns of A, since they are already orthogonal and normalized. The last column becomes

$$\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{q}_1^T\boldsymbol{b}\,\boldsymbol{q}_1 - \boldsymbol{q}_2^T\boldsymbol{b}\,\boldsymbol{q}_2 - \boldsymbol{q}_3^T\boldsymbol{b}\,\boldsymbol{q}_3 = \boldsymbol{b} + \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{q}_4 = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

3 (a) Half of the $n!$ permutations are even and half of them are odd. Multiplying even permutations gives an even permutation. Multiplying odd permutations an odd number of times gives an odd permutation, and an even number of times gives an even permutation. So the problem is equivalent to asking if

$$\frac{n!}{2} = \frac{n \cdot n-1 \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2}$$

is an even or an odd number. The answer is that it is even when $n \geq 4$, it is odd when $n = 2$ or $n = 3$, and in the special case $n = 1$ there is only one even permutation, so that product is even.

(b) You can get the matrix B from A by:

- Adding row 1 and row 2 to row 3.
- Adding row 1 to row 2.
- Exchanging row 1 and row 3.

The first two operations do not change the determinant, and the third changes the sign. Therefore, $\det B = -6$.

(c) (1) For example Property 4: ***If two rows of A are equal, then det A=0***.

(2) Half of the 120 terms are $+1$, and half of them are -1. The sum is zero.