Course 18.06, Fall 2002: Final Exam, Solutions

1 (a)

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}.$$

(b) Four pivots $\Rightarrow$ rank of $A = 4$. The row reduced echolon form is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The two special solutions $(0, 0, 0, -1, 1, 0), (0, 0, 0, -1, 0, 1)$ form a basis for the nullspace.

2 (a) The dimension of the nullspace is 1, so the rank of A is $4 - 1 = 3$. The complete solution to $A\boldsymbol{x} = 0$ is $\boldsymbol{x} = c \cdot (2, 3, 1, 0)$ for any constant c.

(b) The row reduced echelon form has 3 pivots and the special solution $\boldsymbol{x} = (2, 3, 1, 0)$:

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3 (a) The projection matrix P projects onto the column space of P which is the line $c \cdot (1, 2, -4)$.

(b) The vector from $\boldsymbol{b}$ to the subspace is

$$\boldsymbol{e} = \boldsymbol{b} - P\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{21} \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 22 \\ 23 \\ 17 \end{bmatrix}$$

and the distance is

$$\|\boldsymbol{e}\| = \frac{1}{21}\sqrt{22^2 + 23^2 + 17^2} = \frac{\sqrt{1302}}{21}.$$

(c) Since P projects onto a line, its three eigenvalues are $0, 0, 1$. Since P is symmetric, it has a full set of (orthogonal) eigenvectors, and is then diagonalizable.

4 (a) When $AB = 0$, every column of B is in the nullspace of A. So the **null** space of A contains the **column** space of B. Also the **left null** space of B contains the **row** space of A.

(b) The dimension of the nullspace of A is $n - r = 7 - r$. The dimension of the column space of B is s. Since the first contains the second, $7 - r \geq s$, or $r + s \leq 7$.

5 (a) The least squares solution $\hat{\boldsymbol{x}}$ to $Q\boldsymbol{x} = \boldsymbol{b}$ is

$$\hat{\boldsymbol{x}} = (Q^{\mathrm{T}}Q)^{-1}Q^{\mathrm{T}}\boldsymbol{b} = (I)^{-1}Q^{\mathrm{T}}\boldsymbol{b} = Q^{\mathrm{T}}\boldsymbol{b}.$$

(b) One approach: Q^{T} is 2 by 4 so there are free variables and many solutions to $Q^{\mathrm{T}}\boldsymbol{x} = 0$. Then $QQ^{\mathrm{T}}\boldsymbol{x} = 0$ and QQ^{T} is singular; not positive definite.
Second approach: The rank of Q is 2, so the rank of QQ^{T} must be ≤ 2. But QQ^{T} is a 4 by 4 matrix, so the dimension of its nullspace is 2. This means that it has 2 zero eigenvalues, and QQ^{T} is not positive definite.

(c) The singular values of Q are the square roots of the eigenvalues of $Q^{\mathrm{T}}Q = I$, that is, all 1.

6 (a) Let $\boldsymbol{a} = (1, 2, 2)$ and $\boldsymbol{b} = (5, 4, -2)$. The orthonormal vectors are

$$\boldsymbol{q}_1 = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{b} - \frac{\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}}\boldsymbol{q}_1}{\|\cdot\|} = \frac{1}{\sqrt{4^2 + 2^2 + (-4)^2}}\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

where $\|\cdot\|$ means the norm of the numerator.

(b) Let $Q = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix}$. The projection P is then

$$P = QQ^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}$$

(c) The properties of a projection matrix are $P^2 = P$ and $P^{\mathrm{T}} = P$. This gives

$$(P\boldsymbol{b})^{\mathrm{T}}(\boldsymbol{b} - P\boldsymbol{b}) = \boldsymbol{b}^{\mathrm{T}}P^{\mathrm{T}}(\boldsymbol{b} - P\boldsymbol{b}) = \boldsymbol{b}^{\mathrm{T}}(P\boldsymbol{b} - P^2\boldsymbol{b}) = 0.$$

7 (a) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is a basis for $\mathbf{R}^3$ then the matrix with those three columns is **invertible** (non-singular, full rank).

(b) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ span $\mathbf{R}^3$ then the column space is $\mathbf{R}^3$. The only possible rank for the matrix with those four columns is 3.

(c) The transformations of the three basis vectors are

$$\begin{aligned} T(\boldsymbol{q}_1) &= \boldsymbol{q}_1 \\ T(\boldsymbol{q}_2) &= \boldsymbol{q}_2 \\ T(\boldsymbol{q}_3) &= \mathbf{0} \end{aligned}$$

so the transformation matrix T in the basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8

$$\begin{vmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 3 \cdot 27 - 2 \cdot 8 = 65.$$

For general $n > 4$, the determinant is $|A_n| = 3^n + (-1)^{n-1}2^n$.

9 (a) The determinant of a permutation matrix P is 1 or -1. The only possible pivot is 1.

(b) For 3 by 3 permutations, the trace of P is 0, 1, or 3. The eigenvalues are 1 and -1 when two rows are exchanged. Otherwise

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \quad (\text{or } e^{2\pi i/3}, e^{4\pi i/3}),$$

so the four possible eigenvalues are $1, -1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

10 (a)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad A - 3I = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Columns 1, 2, and 4 have pivots. Because of the nonzero bottom-right element in $A - 3I$, the fourth component of $\boldsymbol{x}_3$ is definitely zero.

(b) In the same way as above, the special solutions for the matrices $A - 1I$, $A - 2I$, $A - 3I$, and $A - 4I$ must have 3, 2, 1, and 0 zeros as the last components. The eigenvector matrix S is then upper triangular.