Math 18.06 Quiz 3 Solutions

1 (30 pts.) (a)

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A^{\infty} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) The eigenvalues of B must both be 1. Suppose B has the Jordan form $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, with $B = MJM^{-1}$. Then $B^n = MJ^nM^{-1}$ and $J^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, which cannot converge. So B can NOT have Jordan Form J. The only alternative is that B has Jordan form I, in which case $B = MIM^{-1} = I$

2 (40 pts.)

(a) $S^{-1} = S^T$, so $A = S\Lambda S^T$ is symmetric. Singular values are always nonnegative, so from $\Lambda = \Sigma$ the eigenvalues of A are nonnegative, so A is symmetric positive semidefinite. It can be singular (the all zeros matrix is an example).

(b) The eigenvalues of a projection matrix are either 0 or 1, and their sum is 2, so they must be 1, 1, 0. For example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

(c) $A^TA = \begin{bmatrix} 25 & 0 \\ 0 & 49 \end{bmatrix}$ so the singular values are 7 and 5. So

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{and } A = \begin{bmatrix} 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(d) 1. The eigenvalues of A are 1,1,2 - the same as the eigenvalues of B.
2. A might or might not be diagonalizable
3. A might or might not be symmetric
4. A definitely (!) has positive eigenvalues. However it might not be symmetric, so A might or might not be positive definite.

3 (30 pts.)

(a) The eigenvalues are $0, \sqrt{2}i, -\sqrt{2}i$. They are all pure imaginary (including zero!) because A is skew symmetric

(b) The general solution is $\vec{u}(T) = c_1\vec{x}_1 + c_2 e^{\sqrt{2}iT}\vec{x}_2 + e^{-\sqrt{2}iT}\vec{x}_3$

(c) $e^{i\theta} = \cos\theta + i\sin\theta$. This function has a period of 2π, so when $\sqrt{2}T = 2n\pi$, we have $\vec{u}(T) = \vec{u}(0)$. In particular, T can be $\sqrt{2}\pi$.