Math 18.06 Exam 2 Solutions

1 (36 pts.)

(a) $q_4^* = v - (q_1^T v)q_1 - (q_2^T v)q_2 - (q_3^T v)q_3$

$q_4 = \frac{q_4^*}{\|q_4^*\|}$

(b) The nullspace of Q is just the zero vector (Q has a pivot in every column). The nullspace of Q^T has dimension one and consists of all scalar multiples of q_4 (because we know q_4 is orthogonal to q_1, q_2 and q_3).

The nullspace of $Q^T Q = I$ is just the zero vector. The nullspace of QQ^T again has dimension one and is all scalar multiples of q_4.

(c) $Q^T Q\bar{x} = Q^T b$ is the same as $\bar{x} = Q^T b$, so

$$\bar{x} = \begin{bmatrix} q_1^T(q_1 + 2q_2 + 3q_3 + 4q_4) \\ q_2^T(q_1 + 2q_2 + 3q_3 + 4q_4) \\ q_3^T(q_1 + 2q_2 + 3q_3 + 4q_4) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The projection $p = Q\bar{x} = q_1 + 2q_2 + 3q_3$

2 (24 pts.) (a) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} x = \begin{bmatrix} 3 \\ 5 \\ K \end{bmatrix}$

$Ax = b$ has an exact solution when b is in the column space. This happens when $K = 7$.

(b) $\bar{x} = 0$ is the least squares solution when b is in the nullspace of A^T

For $\begin{bmatrix} 3 \\ 5 \\ K \end{bmatrix}$ to be in the nullspace of A^T, K would have to be -8 and $\frac{-21}{4}$, which is impossible.

3 (40 pts.)

(a) The determinant will have the cofactor of a_{14} added to it. In the second part of the question, the determinant will double.

(b) We know $P^2 = P$, so $(det(P))^2 = det(P)$, so $det(P) = 0$ or 1.

(c) Using cofactors by the first row, $det(C) = (-b)(-b)(a^2-b^2)+(-a)(a)(a^2-b^2) = -(a^2-b^2)^2$

(d) 24 terms using $a_{11}+24$ terms using $a_{22}-6$ terms using both $= 42$ total