18.06 Quiz 3 May 10, 1995 Professor Strang

1. (30 pts) (a) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ -\frac{1}{6} & \frac{1}{6} \end{bmatrix}.$$

(b) Suppose the solution to $u_{k+1} = Au_k$ after 100 steps is $u_{100} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. What was the starting vector u_0? (Same matrix A.)

(c) If B is any other 2 by 2 matrix, explain clearly why AB and BA have the same eigenvalues.

2. (30 pts) Suppose that A is a positive definite matrix:

$$A = \begin{bmatrix} 1 & b & 0 \\ b & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

(a) What are the possible values of b?

(b) How do you know that the matrix $A^2 + I$ is positive definite for *every* b?

(c) Complete this sentence correctly for a general matrix M, possibly rectangular:

The matrix $M^T M$ is symmetric positive definite unless

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3. (32 pts) (a) P is the projection matrix onto the line through $a = (1, 2, 2)$:

$$P = \frac{aa^T}{a^T a} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

What are its eigenvalues? *Describe all of the corresponding eigenvectors.*

(b) Circle *True or False*: There is a matrix S so that $S^{-1}PS$ is a diagonal matrix (and thus P is diagonalized).

(c) Solve the differential equation $\frac{du}{dt} = Pu$ to find $u(t)$ starting from

$$u(0) = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}.$$

(d) For the difference equation $u_{k+1} = Pu_k$ starting from $u_0 = (1, 0, 0)$, what is the vector u_{100}?

4. (8 pts) Give an example of a linear transformation from four-dimensional space R^4 to two-dimensional space R^2.