18.06 Hour Exam II in Linear Algebra

5 November 1993: Professor Strang

Do all your work on these 7 pages. No calculators or notes. Point values (out of a total of 100) are marked on the left margin. This is a long examination!

[25] **1** (a) Find the projection of the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the plane spanned by the vectors $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(b) Apply the Gram-Schmidt process to the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}$ to find orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$.

(c) Express the result of Gram-Schmidt in (b) as a 3×3 matrix factorization $A = QR$.

What is the projection matrix P onto the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

[15] **2.** Which solution of the equations

$$\begin{aligned} x_1 + x_2 + x_3 \qquad\quad &= 0 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

is closest to the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$?

[25] **3.**(a) Find the determinant $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 \\ 2 & 0 & 0 & 7 \end{bmatrix}$. (Hint: Use the cofactor formula.)

Suppose that A is a 3×2 matrix of rank 2. What is $\det(A(A^TA)^{-1}A^T)$?

(c) If A is a non-singular square matrix, what is the third component x_3 of the solution to

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}?$$

[15] **4.** Find the best least squares approximation to the data $0, 1, 3$ at times $-1, 0, 1$, by a curve of the form $At^2 + B$.

[20] **5.** Q is a 4×3 matrix with orthonormal columns $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. Assume that $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, and $\mathbf{b}$ are linearly independent vectors in $\mathbf{R}^4$.

(a) What is the row space of Q?

(b) What combination $\mathbf{p}$ of $\mathbf{q}_1, \mathbf{q}_2$, and $\mathbf{q}_3$ is closest to $\mathbf{b}$?

(c) What combination of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, and $\mathbf{b}$ is in the nullspace of Q^T?