MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1995: Final Examination in 18.06: Linear Algebra

Closed book exam (and no Calculators). Answer all 7 questions in the space provided (or direct us to the answer). Solutions will be posted outside the offices of Professor Kac (2–178) and Professor Axelrod (2–247) who are completely in charge of grades. Best wishes to the whole class.
The first questions are about the symmetric matrices with entries $1, 2, 3, \ldots, n-1$
just above and just below the main diagonal. All other entries are zero:

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \qquad A_5 = \ldots$$

1. (18 points) (a) Find a permutation matrix P_3, a lower triangular L_3 with unit diagonal,
   and an echelon matrix U_3 so that $P_3 A_3 = L_3 U_3$.

1. (b) What is the general (complete) solution to $A_3 x = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$?

1. (c) Give a basis for the left nullspace of A_3. Describe that whole nullspace.

1.(d) Find the projection matrix (call it P) onto the column space of A_3.

2. (12 points) (a) Find the eigenvalues and eigenvectors of A_3.

2. (b) For which initial vectors $u(0)$, if any, will the solution of $\frac{du}{dt} = A_3 u$ decay to zero?

2. (c) Two eigenvalues of A_4 are approximately 3.65 and $.822$. Find the other two eigenvalues using

$$M^{-1} A_4 M = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} = -A_4.$$

3. (12 points) (a) Prove that A_5 is not invertible. $A_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$

3. (b) Is A_5 diagonalizable (similar to a diagonal matrix)? Why or why not?

3. (c) Find the determinant of A_6 (cofactors recommended).

4. (15 points) The least squares solution to

$$Ax = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = b \qquad \text{is} \qquad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 11/3 \\ -1 \end{pmatrix}.$$

4. (a) Find the projection p of b onto the column space of A.

4. (b) Draw the straight-line fit corresponding to this least squares problem. Show on your graph where to see the three components of p.

4. (c) By Gram-Schmidt, find an orthonormal basis q_1, q_2 for the column space of A. *Factor A into QR.* (More space next page.)

5. (12 points) Suppose that the general solution to $Ax = \begin{pmatrix} 3 \\ 1/2 \\ -1 \end{pmatrix}$ is $x = \begin{pmatrix} 5 \\ 11 \\ -9 \\ 7 \end{pmatrix} + t \begin{pmatrix} e \\ \pi \\ 1 \\ 0 \end{pmatrix}$.

   (a) What are the dimensions of $R(A)$ and $N(A)$ and $N\left(A^T\right)$?

   (b) True or False or Undecidable for this A: $Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is solvable.

   Give a reason!

   (c) How do you know that $A^T A$ is not positive definite?

6. (12 points) The column vector $u_k = (R_k, D_k, I_k)$ gives the number of Republicans, Democrats, and Independents in election k. For the next election all Republicans become Independent, while $\frac{1}{3}$ of the Democrats and $\frac{1}{3}$ of the Independents go into each component of $u_{k+1} = (R_{k+1}, D_{k+1}, I_{k+1})$.

   (a) What matrix A gives $u_{k+1} = Au_k$? Check your answer for $u_k = (1, 0, 0)$ and $u_k = (0, 0, 1)$.

   (b) What fractions of the voters are in $R_\infty, D_\infty, I_\infty$ at steady state?

   (c) Find all eigenvalues and eigenvectors of A and find u_k (after k years) if nobody is for Perot at the start:

$$u_0 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

7. (19 points) (a) If you can solve $Ax = b$, then b must be perpendicular to every vector y in the ______________. Give a 3 by 2 example of A and b and y.

7. (b) Find the projection of $b = (1, 2, 2, 7)$ onto the plane $x_1 + x_2 + x_3 + x_4 = 0$. You could project b first onto the line through $a = (1, 1, 1, 1)$.

7. (c) Circle True or False: if A has repeated eigenvalues, it is always possible to find an orthonormal basis for its column space.

7. (d) Circle True or False: if $v_1, \ldots, v_n$ is a basis for R^n and $b = c_1 v_1 + \cdots + c_n v_n$ then $c_1 = \frac{b^T v_1}{b^T b}$.

7. (e) Construct a matrix with eigenvalues $\lambda = 0, 0, 1$ and rank 2. Why can't it be a projection matrix?