The polynomial method for random matrices

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Abstract

We define a class of "algebraically characterizable" random matrices. These are random matrices for which the Stieltjes transform of the limiting spectral measure is an algebraic function. The famous semi-circle law for Wigner matrices and the Marčenko-Pastur law for Wishart matrices are special cases.

The practical utility of this definition can be succinctly summarized: if a random matrix is shown to be algebraic then its limiting spectral measure can be computed using a simple root-finding algorithm. Furthermore, if the moments exist, then the corresponding moment generating function will be differentiably finite so that we will often be able to enumerate them efficiently in closed form.

Algebraicity of a random matrix acts as a certificate of the computability of its limiting spectral measure and moments. We specify the class of such random matrices by its generators and demonstrate that the transforms of "free probability" that encode free additive and multiplicative convolution can be expressed as bivariate resultants. We present a simple computational realization, a random matrix "calculator" as it were, based on the "polynomial method" that finally allows researchers to harness the power of free probability and infinite random matrix theory.

Key words Random matrices, stochastic eigen-analysis, free probability, algebraic functions, resultants, D-finite series.

1. Introduction

We propose a powerful method that allows us to calculate the eigenvalue distributions of a large class of random matrices. We see this method as allowing us to expand our reach beyond the well known special cases such as the semi-circle law [27], the Marčenko-Pastur law [10], the McKay law [11] or their close cousins [4,14]. In particular, we encode transforms of the distribution as roots of a bivariate polynomial equation. Then canonical operations on the random matrices become operations on the bivariate polynomials.

As a simple example, suppose we take the Wigner matrix, generated in MATLAB as:

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G = randn(N)/sqrt(N); A = (G+G')/sqrt(2);
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whose eigenvalues in the $N \to \infty$ limit follow the semicircle law, and the Wishart matrix which may be generated as:

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G = randn(N,2*N)/sqrt(2*N); B = G*G';
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whose eigenvalues in the limit follow the Marčenko-Pastur law. These laws have Stieltjes transforms $m_A(z)$ and $m_B(z)$ that are roots of the equations $L^A_{\mathrm{mz}}(m,z) = 0$ and $L^B_{\mathrm{mz}}(m,z) = 0$, respectively, where $L^A_{\mathrm{mz}}(m,z) \equiv m^2 + z\,m + 1$ and $L^B_{\mathrm{mz}}(m,z) \equiv m^2 z - (-2\,z + 1)\,m + 2$. The sum and product of the random matrices have eigenvalue distribution in the $N \to \infty$ limit whose Stieltjes transform is a root of the bivariate polynomial

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equations $L_{\rm mz}^{A+B}(m,z) = 0$ and $L_{\rm mz}^{AB}(m,z) = 0$, respectively, which can be calculated from $L_{\rm mz}^{A}$ and $L_{\rm mz}^{B}$ alone as shown below.

To compute $L_{\rm mz}^{A+B}$, we use the algorithm in Table 9(c). First, we obtain the polynomials $L_{\rm rg}^{A}(r,g) = r - g$ and $L_{\rm rg}^{B}(r,g) = (g-2)r + 2$ from $L_{\rm mz}^{A}$ and $L_{\rm mz}^{B}$ using the conversions labelled I and II in Table 4. Then we compute $L_{\rm rg}^{A+B}(r,g) \equiv L_{\rm rg}^{A} \boxplus_{\rm r} L_{\rm rg}^{B} = g^2 - (2+r)\,g - 2 + 2\,r$, using the definition of $\boxplus_{\rm r}$ in the upper panel of Table 6. Finally, the polynomial $L_{\rm mz}^{A+B}(m,z) = m^3 + (z+2)\,m^2 - (-2\,z+1)\,m + 2$ can be obtained from $L_{\rm rg}^{A+B}$ using the conversions marked I and II in Table 4.

Similarly, to compute $L_{\rm mz}^{AB}$, we use the algorithm in Table 9(d). First, we obtain the polynomials $L_{\rm sy}^{A}(s,y) = y\,s^2 - 1$ and $L_{\rm sy}^{B}(s,y) = (2+y)\,s - 2$ using the conversion labelled IV in Table 4. Then we compute $L_{\rm sy}^{AB}(s,y) \equiv L_{\rm sy}^{A} \boxtimes_{\rm s} L_{\rm sy}^{B} = \left(4\,y + 4\,y^2 + y^3\right) s^2 - 4$, using the definition of the $\boxtimes_{\rm s}$ operator in the lower panel of Table 6. Finally, the polynomial $L_{\rm mz}^{AB}(m,z) = m^4 z^2 - 2\,m^3 z + m^2 + 4\,mz + 4$ can be obtained from $L_{\rm sy}^{AB}$ using the conversion labelled IV in Table 4.

Figure 1 plots the limiting eigenvalue distribution for the Wigner and Wishart matrices as well as their sum and product. One does not have to stop there; in fact, so long as the limiting density is encoded by a bivariate polynomial one has access to a whole catalog of transformations (see Tables 8 and 9) on random matrices for which the eigenvalue distribution in the $N \to \infty$ limit can be computed from $L_{\rm mz}^{A}$ and $L_{\rm mz}^{B}$ alone.

In this article we demonstrate how by encoding probability densities as bivariate polynomials, and deriving the correct operational laws on this encoding, we can take advantage of powerful symbolic and numerical techniques to compute these densities and their associated moments. In particular, for the example considered, algebraically extracting the roots of these polynomials using the cubic or quartic formulas would be of no use. Our hope in presenting the software version of the catalog operations, encoded as a random matrix "calculator" [12], alongside the mathematics is that readers will take the code as a starting point for their own experimentation and develop additional applications of the theory on which our ideas are based.

2. Motivation

A random matrix is a matrix whose elements are random variables. Let $\mathbf{A}_N$ be an $N \times N$ symmetric/Hermitian random matrix. Its empirical distribution function (e.d.f.) is given by

$$F^{\mathbf{A}_N}(x) = \frac{\text{Number of eigenvalues of } \mathbf{A}_N \leq x}{N}. \tag{2.1}$$

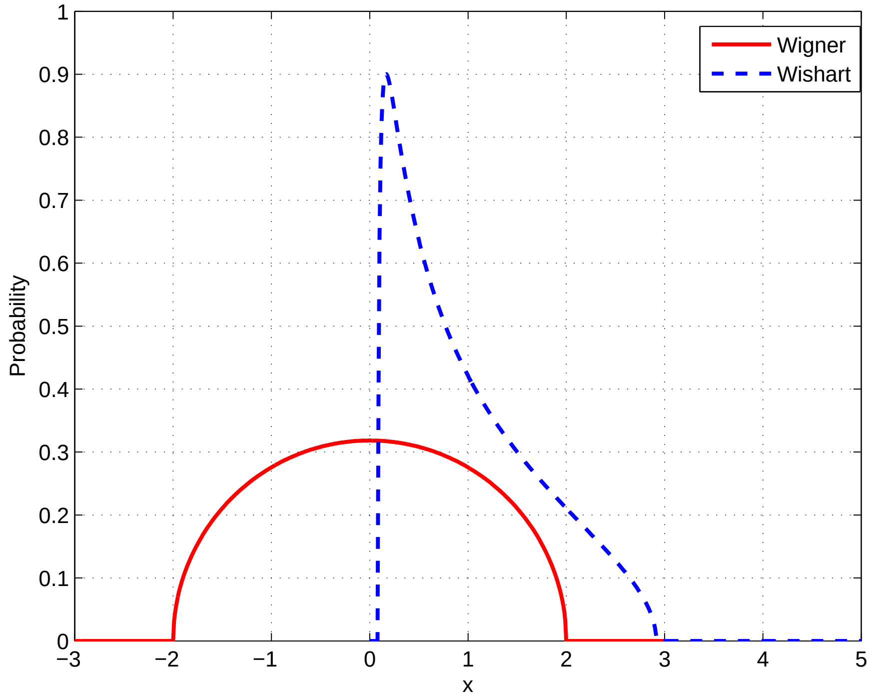
We are interested in the associated (discrete or normalized counting) probability measure

$$f_{\mathbf{A}_N}(x) := \frac{1}{N} \sum_{i=1}^{N} \delta(x - \lambda_i), \tag{2.2}$$

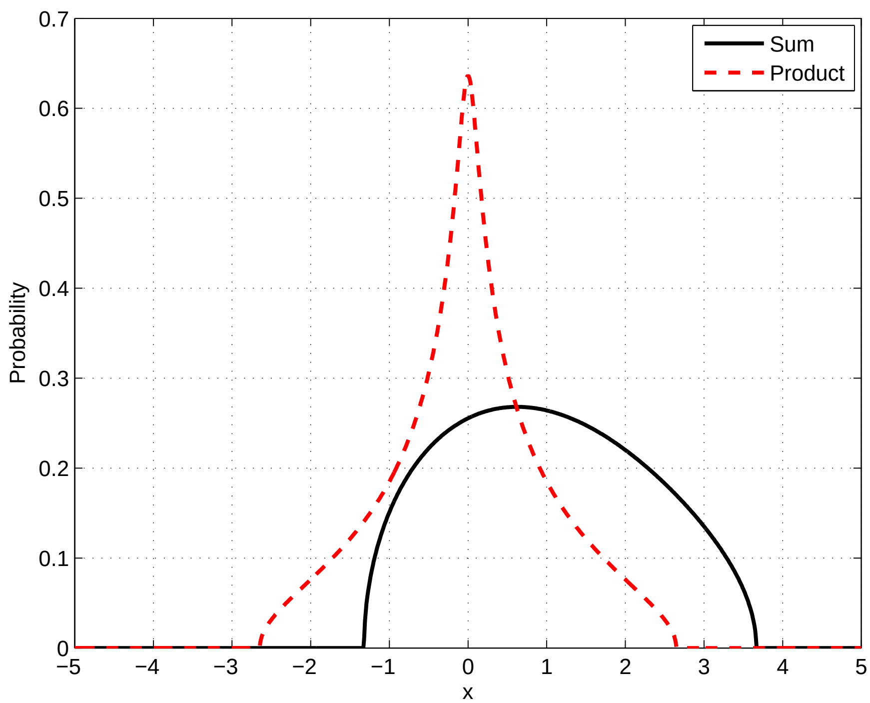
where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues (counting multiplicities) of $\mathbf{A}_N$ and $\delta(\cdot)$ is the Dirac delta function.

For a large class of random matrices, the empirical distribution function $F^{\mathbf{A}_N}$ converges in distribution almost surely (or in probability) as $N \to \infty$ to a non-random distribution function F^A. Denote the corresponding spectral measure by f_A. This predicted spectral measure tends to very well approximate the eigenvalue distribution of "large enough" random matrices. In other words, the histogram of the eigenvalues of a random matrix normalized to have area 1 when averaged over many trials or for one large random matrix will be well approximated by this limiting spectral measure.

The limiting spectral measure can be encoded using the transforms listed in Table 1. The Cauchy/Stieltjes transform is the most fundamental. The R and S transforms play a role in the computation of the limiting spectral measure of the sum and product of random matrices respectively. The moment and eta transforms are minor variations of the Stieltjes transform. Canonical random matrix operations can be expressed explicitly as transformations of the first four transforms. We encode these implicitly by deriving the operational laws on the appropriate bivariate polynomials.



(a) The limiting spectral measures of the Wigner and Wishart matrices.



(b) The limiting spectral measures of the sum and product of independent Wigner and Wishart matrices.

Figure 1: A representative computation using the random matrix calculator.

3. Our Algebraic Framework

Notation 3.1 (Bivariate polynomial). *Let L_{uv} denote a bivariate polynomial of degree $\mathrm{D_u}$ in u and $\mathrm{D_v}$ in v:*

$$L_{\mathrm{uv}} \equiv L_{\mathrm{uv}}(\cdot,\cdot) = \sum_{j=0}^{\mathrm{D_u}} \sum_{k=0}^{\mathrm{D_v}} c_{jk}\, u^j\, v^k = \sum_{j=0}^{\mathrm{D_u}} l_j(v)\, u^j. \tag{3.1}$$

The scalar coefficients c_{jk} are real valued.

Remark 3.2 (Irreducibility). *Unless otherwise stated it will be understood that $L_{\mathrm{uv}}(u,v)$ is "irreducible" in the sense that the conditions:*

- $l_0(v), \ldots, l_{\mathrm{D_u}}(v)$ *have no common factor involving v,*
- $l_{\mathrm{D_u}}(v) \neq 0$,
- $\mathrm{disc_L}(v) \neq 0$,

are satisfied, where $\mathrm{disc}_L(v)$ is the discriminant of $L_{\mathrm{uv}}(u,v)$ thought of as a polynomial in v.

We are particularly focused on the root "curves," $u_1(v), \ldots, u_{\mathrm{D_u}}(v)$, i.e.,

$$L_{\mathrm{uv}}(u,v) = l_{\mathrm{D_u}}(v) \prod_{i=1}^{D_{\mathrm{u}}} (u - u_i(v))\,.$$

Informally speaking, when we refer to the bivariate polynomial L_{uv} with roots $u_i(v)$ we are actually considering the equivalence class of rational functions with this set of root curves.

Remark 3.3 (Equivalence class). *The equivalence class of $L_{\mathrm{uv}}(u,v)$ may be characterized as functions of the form $L_{\mathrm{uv}}(u,v)g(v)/h(u,v)$ where h is relatively prime to $L_{\mathrm{uv}}(u,v)$ and $g(v)$ is not identically* 0.

A few technicalities (such as poles and singular points) that will be catalogued later in Section 11. remain, but this is sufficient for allowing us to introduce rational transformations of the arguments and continue to use the language of polynomials.

Definition 3.4 (Atomic probability density). *Let $f(x)$ be a probability measure of the form:*

$$f(x) = \sum_{i=1}^{K} p_i\, \delta(x - \lambda_i),$$

where the K atoms at $\lambda_i \in \mathbb{R}$ have (non-negative) weights p_i subject to $\sum_i p_i = 1$. We refer to $f(x)$ as an atomic probability density or atomic probability measure and say that $f(x) \in \mathcal{P}_{\mathrm{atom}}$. Here $\mathcal{P}_{\mathrm{atom}}$ denotes the class of atomic probability measures.

Remark 3.5. *The probability measure in (2.2) is an atomic probability density.*

Definition 3.6 (Algebraic density). *Let $f(x)$ be a probability measure. Consider the Stieltjes transform, $m(z)$, which is a complex valued function defined on $z \in \mathbb{C} \setminus \mathbb{R}$ as:*

$$m(z) = \int \frac{1}{x-z} f(x) dx \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.2}$$

If there exists a bivariate polynomial L_{mz} such that $L_{\text{mz}}(m(z), z) = 0$ then we refer to $f(x)$ as an algebraic density or algebraic probability measure and say the $f \in \mathcal{P}_{alg}$. Here $\mathcal{P}_{\text{alg}}$ denotes the class of algebraic probability measures.

Notation 3.7 (Region of support). *Let $\mathcal{S}_A$ denote the support of $f_A(x)$. Let $\mathcal{S}_A^{\delta}$ and $\mathcal{S}_A^{-}$ denote the atomic and non-atomic components of the support, respectively. Thus $\mathcal{S}_A = \mathcal{S}_A^{\delta} \cup \mathcal{S}_A^{-}$.*

Definition 3.8 (Positive semi-definite algebraic density). *Algebraic densities supported over the positive real axis are referred to as positive semi-definite. We denote the class of positive semi-definite densities by $\mathcal{P}_{\text{alg}}^{+}$. Thus, if $\mathcal{S}_A \subset [0, \infty)$ then we say that $f_A(x) \in \mathcal{P}_{\text{alg}}^{+}$.*

Some examples of algebraic densities follow.

Example 3.9. *An atomic probability density is an algebraic density since its Stieltjes transform is of the form:*

$$m(z) = \sum_{i=1}^{K} \frac{p_i}{\lambda_i - z}.$$

Example 3.10. *The Cauchy distribution, which is the probability measure:*

$$f(x) = \frac{1}{\pi(x^2+1)},$$

has a Stieltjes transform $m(z)$ which is the zero of $L_{\text{mz}}(m,z) \equiv \left(z^2+1\right) m^2 + 2\, z\, m + 1$. Hence it is an algebraic density.

It is often the case that algebraic densities, according to our definition, will also be algebraic functions themselves. We conjecture that this is a necessary but not sufficient condition. We show that it is not sufficient by providing the counter-example below.

Counter-example 3.11. *The quarter circle law is the probability measure:*

$$f(x) = \frac{\sqrt{4-x^2}}{\pi} \qquad \text{for } x \in [0,2].$$

Its Stieltjes transform given by:

$$m(z) = -\frac{4 - 2\sqrt{-z^2+4}\ln\left(-\frac{2+\sqrt{-z^2+4}}{z}\right) + z\pi}{2\pi},$$

is clearly not an algebraic function. Thus $f(x) \notin \mathcal{P}_{\text{alg}}$.

Let $\mathbf{A}_N$, for $N = 1, 2, \ldots$ be a sequence of $N \times N$ random matrices with real eigenvalues. Let $F^{\mathbf{A}_N}$ denote the e.d.f., as in (2.1). Suppose $F^{\mathbf{A}_N}$ converges weakly to F^A almost surely (or in probability) as $N \to \infty$, then we say that $\mathbf{A}_N \to A$. Denote the corresponding limiting probability measure by f_A.

Notation 3.12 (Mode of convergence). *When necessary we highlight the mode of convergence thus: if* $\mathbf{A}_N \xrightarrow{a.s.} A$ *then the convergence (in distribution) is almost surely; if* $\mathbf{A}_N \xrightarrow{p} A$ *then the convergence (in distribution) is in probability. When the distinction is not made then it is to be assumed that the results hold for both modes of convergence.*

Remark 3.13. *The element A above is not to be interpreted as a matrix. There is no convergence in the sense of an $\infty \times \infty$ matrix. We associate with each A, a probability measure f_A.*

Definition 3.14 (Atomic random matrix). *If $f_A \in \mathcal{P}_{\text{atom}}$ then we say that $\mathbf{A}_N$ is an atomic random matrix. We represent this as $\mathbf{A}_N \to A \in \mathcal{M}_{\text{atom}}$ where $\mathcal{M}_{\text{atom}}$ denotes the class of atomic random matrices.*

Definition 3.15 (Algebraic random matrix). *If $f_A \in \mathcal{P}_{\text{alg}}$ then we say that $\mathbf{A}_N$ is an algebraically characterizable random matrix (often suppressing the word characterizable for brevity). We represent this as $\mathbf{A}_N \to A \in \mathcal{M}_{\text{alg}}$ where $\mathcal{M}_{\text{alg}}$ denotes the class of algebraic random matrices. Note that by definition, $\mathcal{M}_{\text{atom}} \subset \mathcal{M}_{\text{alg}}$.*

Remark 3.16. *When we label a random matrix $\mathbf{A}_N$ as algebraic, it is to be understood that we are characterizing the sequence of random matrices $\mathbf{A}_N$.*

Remark 3.17. *It will be assumed that the random matrix transformations operate on and produce well-defined symmetric/hermitian matrices with real eigenvalues. The conditions will be obvious from context. Thus, for example, the matrix $\mathbf{C}_N = \mathbf{A}_N \times \mathbf{B}_N$ makes sense only if $\mathbf{A}_N$ or $\mathbf{B}_N$ or both are positive semi-definite so that $\mathbf{C}_N = \mathbf{A}_N^{1/2}\mathbf{B}_N\mathbf{A}_N^{1/2}$ or $\mathbf{C}_N = \mathbf{B}_N^{1/2}\mathbf{A}_N\mathbf{B}_N^{1/2}$. Here $\mathbf{A}_N^{1/2}$ denotes the symmetric square root of $\mathbf{A}_N$.*

Remark 3.18 (Proof by construction). *In this article we prove existence of the bivariate polynomial by construction. Thus, if $\mathbf{A}_N \to A \in \mathcal{M}_{\text{alg}}$, we assume that L^A_{mz} is known.*

4. Main Results

The ability to describe the class of algebraic random matrices and the technique needed to compute the associated bivariate polynomial is at the crux our investigation. In the theorems that follow, we accomplish the former by cataloguing random matrix operations that preserve algebraicity of the limiting spectral measure.

4.1 Deterministic transformations

We first consider some simple deterministic transformations on an algebraic random matrix $\mathbf{A}_N$ that produce an algebraic random matrix $\mathbf{B}_N$.

Theorem 4.1. *Let* $\mathbf{A}_N \to A \in \mathcal{M}_{\text{alg}}$. *Then,*

1. $\mathbf{B}_N = (p\,\mathbf{A}_N + q\,\mathbf{I}_N)/(r\,\mathbf{A}_N + s\,\mathbf{I}_N) \to B \in \mathcal{M}_{\text{alg}}$, *if* $\{-s/r\} \notin \mathcal{S}_A^\delta$ *(see Notation 3.7),*
2. $\mathbf{A}_N = \mathbf{X}_{n,N}\mathbf{X}_{n,N}^{'}$ *then* $\mathbf{B}_N = \mathbf{X}_{n,N}^{'}\mathbf{X}_{n,N} \to B \in \mathcal{M}_{\text{alg}}$,
3. $\mathbf{B}_N = \text{diag}(\mathbf{A}_n, \alpha\,\mathbf{I}_{N-n}) \to B \in \mathcal{M}_{\text{alg}}$,
4. $\mathbf{A}_n = \text{diag}(\mathbf{B}_N, \alpha\,\mathbf{I}_{n-N})$*;* $\mathbf{B}_N \to B \in \mathcal{M}_{\text{alg}}$.
5. $\mathbf{B}_N = (\mathbf{A}_N)^2 \to B \in \mathcal{M}_{\text{alg}}$,

Here α, p, q, r, *and* s *are real-valued scalars,* $n/N \to c > 0$ *as* $n, N \to \infty$ *and* $\mathbf{I}_N$ *is the* $N \times N$ *identity matrix.*

Theorem 4.1.1 leads to the useful corollary below.

Corollary 4.1. *Let* $\mathbf{A}_N \to A \in \mathcal{M}_{\text{alg}}$. *Then,*

1. $\mathbf{B}_N = \mathbf{A}_N^{-1} \to B \in \mathcal{M}_{\text{alg}}$, *if* $\{0\} \notin \mathcal{S}_A^\delta$
2. $\mathbf{B}_N = \alpha\,\mathbf{A}_N \to B \in \mathcal{M}_{\text{alg}}$,
3. $\mathbf{B}_N = \mathbf{A}_N + \alpha\,\mathbf{I}_N \to B \in \mathcal{M}_{\text{alg}}$.

A simple extension of Theorem 4.1.3 shows that a block diagonal random matrix $\mathbf{C}_N$ constructed from two algebraic random matrices $\mathbf{A}_N$ and $\mathbf{B}_N$ is also algebraic.

Theorem 4.2. *Let* $\mathbf{A}_N \to A \in \mathcal{M}_{\text{alg}}$ *and* $\mathbf{B}_N \to B \in \mathcal{M}_{\text{alg}}$. *Then,* $\mathbf{C}_M = \text{diag}(\mathbf{A}_n, \mathbf{B}_N) \to C \in \mathcal{M}_{\text{alg}}$. *Here* $M = n + N$ *and* $n/N \to c > 0$ *as* $n, N \to \infty$.

4.2 Stochastic transformations

Definition 4.3 (Gaussian-like random matrix). *Let* $\mathbf{Y}_{N,L}$ *be an* $N \times L$ *matrix with independent, identically distributed (i.i.d.) elements having zero mean, unit variance and bounded higher order moments. We label the matrix* $\mathbf{G}_{N,L} = \frac{1}{\sqrt{L}}\mathbf{Y}_{N,L}$ *as a Gaussian-like random matrix.*

We can generate a Gaussian-like random matrix in MATLAB as `G = sign(randn(N/L))/sqrt(L)`. Gaussian-like matrices are labelled thus because they exhibit the same limiting behavior in the $N \to \infty$ limit as "pure" Gaussian matrices generated as `G = randn(N/L)/sqrt(L)`.

Definition 4.4 (Wishart-like random matrix). *Let $\mathbf{G}_{N,L}$ be a Gaussian-like random matrix. We label the matrix $W_N = \mathbf{G}_{N,L} \times \mathbf{G}'_{N,L}$ as a Wishart-like random matrix. Let $c_N = N/L$. We denote a Wishart-like random matrix thus formed by $\mathbf{W}_N(c_N)$. The limiting spectral measure of the Wishart-like random matrix is the Marčenko-Pastur law which is an algebraic density.*

We now consider some simple stochastic transformations that "blur" the eigenvalues of $\mathbf{A}_N$ by injecting additional randomness. We show that canonical operations involving an algebraic random matrix, $\mathbf{A}_N$, and Gaussian-like and Wishart-like random matrices produce an algebraic random matrix $\mathbf{B}_N$.

Theorem 4.5. *Let $\mathbf{A}_N \to A \xrightarrow{\text{a.s.}} \mathcal{M}_{\text{alg}}$. Then,*

1. $\mathbf{B}_N = \mathbf{A}_N + \mathbf{G}^{'}_{L,N}\mathbf{T}_L\mathbf{G}_{L,N} \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$, *if* $\mathbf{T}_L \xrightarrow{\text{a.s.}} T \in \mathcal{M}_{\text{atom}}$,

2. $\mathbf{B}_N = \mathbf{A}_N \times \mathbf{W}_N(c_N) \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$, *if* $f_A \in \mathcal{P}^{+}_{\text{alg}}$,

3. $\mathbf{B}_N = (\mathbf{A}_N^{1/2} + \sqrt{s}\,\mathbf{G}_{N,L})(\mathbf{A}_N^{1/2} + \sqrt{s}\,\mathbf{G}_{N,L})' \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$, *if* $f_A \in \mathcal{P}^{+}_{\text{alg}}$.

where $\mathbf{A}_N^{1/2}$ denotes a an $N \times L$ matrix such that $(\mathbf{A}_N^{1/2})(\mathbf{A}_N^{1/2})' = \mathbf{A}_N$. Here s is a non-negative real-valued scalar and $c_N = N/L \to c > 0$ as $N, L \to \infty$.

Given two algebraic random matrices $\mathbf{A}_N$ and $\mathbf{B}_N$, we can add and multiply them, provided we spin around the eigenvectors of one of these matrices randomly, to obtain an algebraic random matrix $\mathbf{C}_N$.

Theorem 4.6. *Let $\mathbf{A}_N \xrightarrow{\text{p}} A \in \mathcal{M}_{\text{alg}}$, $\mathbf{B}_N \xrightarrow{\text{p}} B \in \mathcal{M}_{\text{alg}}$ and $\mathbf{Q}_N$ is an $N \times N$ Haar orthogonal/unitary random matrix independent of $\mathbf{A}_N$ and $\mathbf{B}_N$. Then,*

1. $\mathbf{C}_N = \mathbf{A}_N + \mathbf{Q}_N\mathbf{B}_N\mathbf{Q}^{'}_N \xrightarrow{\text{p}} C \in \mathcal{M}_{\text{alg}}$

2. $\mathbf{C}_N = \mathbf{A}_N \times \mathbf{Q}_N\mathbf{B}_N\mathbf{Q}^{'}_N \xrightarrow{\text{p}} C \in \mathcal{M}_{\text{alg}}$

Here multiplication makes sense only if the resulting matrix has real eigenvalues.

When both f_A and f_B in Theorem 4.6 are compactly supported, it is possible to strengthen the mode of convergence to almost surely [9]. With careful analysis, we believe that it should be possible to derive almost sure convergence when either or both of the densities are non-compact.

Definition 4.7 (Orthogonally/Unitarily invariant random matrix). *If the joint distribution of the elements of a random matrix $\mathbf{A}_N$ is invariant under orthogonal/unitary transformations, it is referred to as an orthogonally/unitarily invariant random matrix.*

If $\mathbf{A}_N$ (or $\mathbf{B}_N$) or both are an orthogonally/unitarily invariant sequence of random matrices then Theorem 4.6 can be stated more simply.

Corollary 4.2. *Let* $\mathbf{A}_N \xrightarrow{\text{p}} A \in \mathcal{M}_{\text{alg}}$ *and* $\mathbf{B}_N \to B \xrightarrow{\text{p}} \mathcal{M}_{\text{alg}}$ *be a orthogonally/unitarily invariant random matrix independent of* $\mathbf{A}_N$*. Then,*

1. $\mathbf{C}_N = \mathbf{A}_N + \mathbf{B}_N \xrightarrow{\text{p}} C \in \mathcal{M}_{\text{alg}}$

2. $\mathbf{C}_N = \mathbf{A}_N \times \mathbf{B}_N \xrightarrow{\text{p}} C \in \mathcal{M}_{\text{alg}}$

Here multiplication makes sense only if the resulting matrix has real eigenvalues.

In the above corollary, the limiting spectral measure of $\mathbf{C}_N$ is computed from the limiting spectral measure of $\mathbf{A}_N$ and $\mathbf{B}_N$ by free additive and multiplicative convolution respectively. Thus we may readily infer the properties of algebraic densities have under free additive and multiplicative convolution.

Corollary 4.3. *Algebraic densities form a semi-group under free additive convolution.*

Corollary 4.4. *Positive semi-definite algebraic densities form a semi-group under free multiplicative convolution.*

5. The polynomial method and algebraic varieties

For a canonical random matrix operation, the computation of the transformed bivariate polynomial L_{mz} relies on manipulations between various bivariate polynomials. In this article, the important interconnected bivariate polynomials will turn out to be:

- $L_{\text{mz}}(m, z)$, where $m(z)$ is the Stieltjes transform,
- $L_{\text{rg}}(r, g)$, where $r(g)$ is the R transform and,
- $L_{\text{sy}}(s, y)$, where $s(y)$ is the S transform.

Each of the transforms encode information about the limiting spectral measure we are interested in. The bivariate polynomials define each transform implicitly via the equation $L_{\text{uv}}(u, v) = 0$. Moreover, each bivariate polynomial, say L_{mz}, can be obtained from another bivariate polynomial, say L_{rg}, by a very simple change of variables. The transformations between the polynomials, summarized in Table 4, is an important component of our technique.

Definition 5.1 (Algebraic variety for algebraic random matrices). *In algebraic terms, given polynomials* L_{mz}*,* L_{rg}*, and* L_{sy}*, the algebraic variety,* $\mathcal{V}_{\text{alg}}$*, is defined as the set:*

$$\mathcal{V}_{\text{alg}}(L_{\text{mz}}, L_{\text{rg}}, L_{\text{sy}}) := \{\mathbf{a} = [m, z, r, g, s, y]^T \in \mathbb{C}^6 \,|\, L_{\text{mz}}(m, z) = L_{\text{rg}}(r, g) = L_{\text{sy}}(s, y) = 0\}. \tag{5.1}$$

Example 5.2 (Algebraic variety for the Wigner matrix). *The limiting spectral measure of the Wigner matrix is given by the semi-circular law:*

$$f_A(x) = \frac{\sqrt{4-x^2}}{2\,\pi} \qquad \text{for } x \in [-2, 2]. \tag{5.2}$$

The Stieltjes transform, $m(z)$, of this density is a zero of the bivariate polynomial $L^A_{\text{mz}} \equiv m^2 + m\,z + 1$. The R transform $r(g)$ is a zero of the bivariate polynomial $L^A_{\text{rg}} \equiv r - g$ while the S transform $s(y)$ is a zero of the bivariate polynomial $L^A_{\text{sy}} \equiv s^2\,y - 1$. The bivariate polynomial equations define an algebraic variety as in (5.1).

Canonical random matrix transformations (summarized in Tables 8 9) are mapped into transformations of the algebraic variety. This transformation, though not explicitly described in terms of the variety, involves a sequence of steps that are captured in the example considered. First, we compute the projection onto $\mathbb{C}^2$ for the (m, z) or (r, g) or (s, y) coordinates. Then we compute the transformation of the appropriate bivariate polynomial in this coordinate system. Finally, use the relationship between the bivariate polynomials to compute the algebraic variety defined in $\mathbb{C}^6$.

The key idea is that some random matrix transformations, say $\mathbf{A}_N \longmapsto \mathbf{B}_N$, can be most naturally expressed as transformations of $L^A_{\text{mz}} \longmapsto L^B_{\text{mz}}$; others as $L^A_{\text{rg}} \longmapsto L^B_{\text{rg}}$ while some as $L^A_{\text{sy}} \longmapsto L^B_{\text{sy}}$. Hence we manipulate the bivariate polynomials to the form needed to apply the appropriate operational law (which we shall derive) and then reverse the transformations to obtain the bivariate polynomial L^B_{mz}. Once we have L^B_{mz} the spectral measure and the associated moments can be readily computed.

6. Transform representations

6.1 The Stieltjes transform and some minor variations

The Stieltjes transform of the spectral measure $f_A(x)$ is given by

$$m_A(z) = \int \frac{1}{x-z} f_A(x)dx \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{6.1}$$

The Stieltjes transform as in (6.1) is well-defined for values of $z \in \mathbb{C} \setminus \mathcal{S}_A$; particularly, it is real valued for $z \in \mathbb{R} \setminus \mathcal{S}_A$. If there are any atomic components in the region of support, i.e. $\mathcal{S}^\delta_A \neq \emptyset$, the Stieltjes transform will have a pole at those values of $z \in \mathcal{S}^\delta_A \subset \mathbb{R}$. The Stieltjes transform is discontinuous at other values of $z \in \mathcal{S}^-_A$ though its real part is continuous and given by a Cauchy principal value integral. This discontinuity is a consequence of the fact that

$$\lim_{\xi \to 0^+} \text{Im}\ m_A(z + i\xi) = -\lim_{\xi \to 0^+} \text{Im}\ m_A(z - i\xi) \qquad \text{for } z \in \mathcal{S}^-_A$$

i.e. the Stieltjes transform is "double valued" in the region of support. The Stieltjes transform can also be interpreted as an expectation with respect to the spectral measure $f_A(x)$ such that

$$m_A(z) = E_X\left[\frac{1}{x-z}\right].$$

A natural consequence of this interpretation is that for any invertible function $h(x)$ continuous over $\mathcal{S}_A$, the Stieltjes transform $m_A(z)$ can also be written in terms of the random variable $y = h(x)$ as:

$$m_A(z) = E_X\left[\frac{1}{x-z}\right] = E_Y\left[\frac{1}{h^{\langle -1\rangle}(y) - z}\right] \tag{6.2}$$

where $h^{\langle -1\rangle}(x)$ is the inverse of $h(x)$ with respect to composition i.e. $h(h^{\langle -1\rangle}(x)) = x$. Equivalently, the expectation with respect to the measure on y can be expressed in terms of the measure on x as

$$E_Y\left[\frac{1}{y-z}\right] = E_X\left[\frac{1}{h(x)-z}\right]. \tag{6.3}$$

The well-known Stieltjes-Perron inversion formula [1] can be used to recover the probability measure from the Stieltjes transform:

$$f_A(x) = \frac{1}{\pi}\lim_{\xi\to 0^+} \Im\, m_A(x+i\xi). \tag{6.4}$$

In a portion of the literature on random matrices, the Cauchy transform is defined as:

$$g_A(z) = \int \frac{1}{z-x} f_A(x)dx.$$

The Cauchy transform is identical to the Stieltjes transform, as defined in (6.1), except for a sign change:

$$g_A(z) = -m_A(z). \tag{6.5}$$

6.2 The moment transform

When the spectral measure $f_A(x)$ is compactly supported, the Stieltjes transform can also be expressed as a "multipole" series expansion analytic about $z=\infty$ as:

$$m_A(z) = -\int_{\mathcal{S}_A}\sum_{j=0}^{\infty}\frac{x^j}{z^{j+1}} f_A(x)dx = -\frac{1}{z} - \sum_{j=1}^{\infty}\frac{M_j^A}{z^{j+1}}, \tag{6.6}$$

where M_j^A are the associated moments. The ordinary moment generating function, $\mu_A(z)$, is a power series analytic about $z=0$ defined as

$$\mu_A(z) = \sum_{j=0}^{\infty} M_j^A\, z^j, \tag{6.7}$$

where $M_0^A \equiv 1$. The moment generating function, referred to as the moment transform, is related to the Stieltjes transform as:

$$\mu_A(z) = -\frac{1}{z} m_A\left(\frac{1}{z}\right). \tag{6.8}$$

Equivalently, the Stieltjes transform can be written in terms of the moment transform as:

$$m_A(z) = -\frac{1}{z}\mu_A\left(\frac{1}{z}\right). \tag{6.9}$$

The eta transform, introduced by Tulino and Verdù in [22], is a minor variation of the moment transform. It and can be written in terms of the Stieltjes transform as:

$$\eta_A(z) = \frac{1}{z} m_A\left(-\frac{1}{z}\right), \tag{6.10}$$

while the Stieltjes transform can be written in terms of the eta transform as:

$$m_A(z) = -\frac{1}{z}\eta_A\left(-\frac{1}{z}\right). \tag{6.11}$$

6.3 The R transform

The R transform is defined in terms of the Cauchy transform as:

$$r_A(z) = g_A^{\langle -1 \rangle}(z) - \frac{1}{z}, \tag{6.12}$$

where $g_A^{\langle -1 \rangle}(z)$ is the functional inverse of $g_A(z)$ with respect to composition. It will often be more convenient to use the expression for the R transform in terms of the Cauchy transform given by:

$$r_A(g) = z(g) - \frac{1}{g}. \tag{6.13}$$

The R transform can be written as a power series whose coefficients, K_j^A, are known as the "free cumulants". For a combinatorial interpretation of free cumulants, see [18]. Thus the R transform is the (ordinary) free cumulant generating function:

$$r_A(g) = \sum_{j=1}^{\infty} K_j^A \, g^{j-1}. \tag{6.14}$$

6.4 The S transform

The S transform is relatively more complicated. It is defined as:

$$s_A(z) = \frac{1+z}{z} \, \Upsilon_A^{\langle -1 \rangle}(z) \tag{6.15}$$

where $\Upsilon_A(z)$ can be written in terms of the Stieltjes transform $m_A(z)$ as:

$$\Upsilon_A(z) = -\frac{1}{z} m_A(1/z) - 1. \tag{6.16}$$

This definition is quite cumbersome to work with because of the functional inverse in (6.15). It also places a technical restriction (to enable series inversion) that $M_1^A \neq 0$. We can, however, avoid this by expressing the S transform algebraically in terms of the Stieltjes transform as shown next. We first plug in $\Upsilon_A(z)$ into the left-hand side of (6.15) to obtain

$$s_A(\Upsilon_A(z)) = \frac{1 + \Upsilon_A(z)}{\Upsilon_A(z)} \, z.$$

This can be rewritten in terms of $m_A(z)$ using the relationship in (6.16) as:

$$s_A(-\frac{1}{z} m(1/z) - 1) = \frac{z \, m(1/z)}{m(1/z) + z}$$

or, equivalently:

$$s_A(-z \, m(z) - 1) = \frac{m(z)}{z \, m(z) + 1}. \tag{6.17}$$

We now define $y(z)$ in terms of the Stieltjes transform as $y(z) = -z \, m(z) - 1$. It is clear that $y(z)$ is an invertible function of $m(z)$. The right hand side of (6.17) can be rewritten in terms of $y(z)$ as:

$$s_A(y(z)) = -\frac{m(z)}{y(z)} = \frac{m(z)}{z \, m(z) + 1}. \tag{6.18}$$

Equation (6.18) can be rewritten to obtain a simple relationship between the Stieltjes transform and the S transform:

$$m_A(z) = -y \, s_A(y) \tag{6.19}$$

Noting that $y = -z\,m(z) - 1$ and $m(z) = -y\,s_A(y)$ we obtain that:

$$y = z\,y\,s_A(y) - 1$$

or, equivalently

$$z = \frac{y+1}{y\,s_A(y)}. \tag{6.20}$$

7. Bivariate polynomials

We define more formally six interconnected bivariate polynomials denoted by L_{mz}, L_{gz}, L_{rg}, L_{sy}, $L_{\mu\text{z}}$, and $L_{\eta\text{z}}$. We assume that $L_{\text{uv}}(u,v)$ is a bivariate polynomial of the form in (3.1). The main protagonist of the transformations we describe in this article is the bivariate polynomial L_{mz} which implicitly defines the Stieltjes transform $m(z)$ via the equation $L_{\text{mz}}(m,z) = 0$. Starting off with this polynomial we can obtain the polynomial L_{gz} using the relationship in (6.5) as

$$L_{\text{gz}}(g,z) = L_{\text{mz}}(-g,z). \tag{7.1}$$

Perhaps we should explain our abuse of notation. Given any one polynomial, all six polynomials exist. The two letter subscripts not only tell us which of the six polynomials we are focusing on, it provides a convention of which dummy variables we will use. The first letter in the subscript represents the transform; the second letter is a mnemonic for the variable associated with the transform that we use consistently in our software and this article. With this notation in mind, we can obtain the polynomial L_{rg} from L_{gz} using (6.13) as:

$$L_{\text{rg}}(r,g) = L_{\text{gz}}\left(g, r + \frac{1}{g}\right). \tag{7.2}$$

Similarly, we can obtain the bivariate polynomial L_{sy} from L_{mz} using the expressions in (6.19) and (6.20) to obtain the relationship

$$L_{\text{sy}} = L_{\text{mz}}\left(-y\,s, \frac{y+1}{sy}\right). \tag{7.3}$$

Based on the transforms discussed in Section 6. we can derive transformations between additional pairs of bivariate polynomials depicted by the bidirectional arrows in Figure 2 and listed in the third column of Table 4. Specifically, the expressions in (6.8) and (6.11) can be used to derive the transformations between L_{mz} and $L_{\mu\text{z}}$ and L_{mz} and $L_{\eta\text{z}}$ respectively. The fourth column of Table 4 lists the MATLAB function, implemented using its Maple based Symbolic Toolbox, corresponding to the bivariate polynomial transformations depicted in Figure 2. In the MATLAB functions, the function `irreducLuv(u,v)` listed in Table 7. that ensures that the resulting bivariate polynomial is irreducible by clearing the denominator and making the resulting polynomial square free.

Example: Let $f(x)$ be an atomic probability measure:

$$f(x) = 0.5\,\delta(x) + 0.5\,\delta(x-1). \tag{7.4}$$

The Stieltjes transform of $f(x)$ is:

$$m(z) = \frac{0.5}{0-z} + \frac{0.5}{1-z},$$

which can be written as the solution to the equation:

$$m(0-z)(1-z) - 0.5(1-2z) = 0,$$

thereby allowing us to identify the bivariate polynomial $L_{\text{mz}}(m,z)$ as simply:

$$L_{\text{mz}}(m,z) \equiv m(2\,z - 2\,z^2) - (1-2z). \tag{7.5}$$

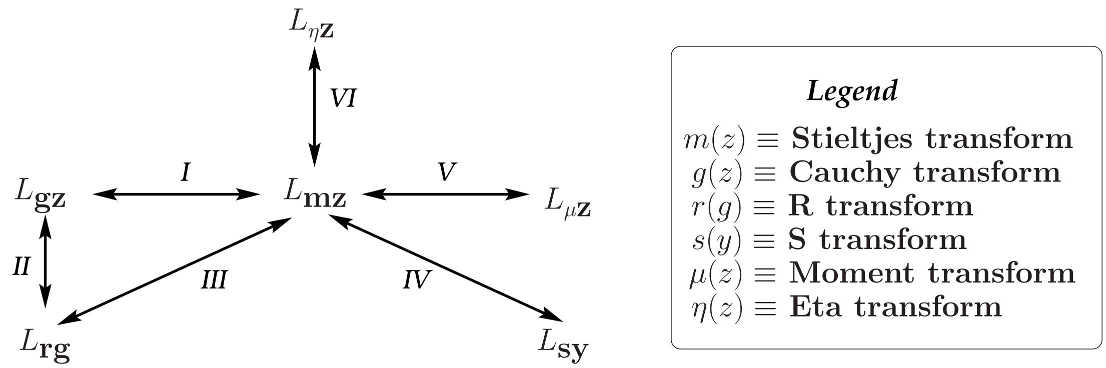


Figure 2: The six interconnected bivariate polynomials; transformations between the polynomials, indicated by the labelled arrows, are given in Table 4.

We can obtain the bivariate polynomial $L_{\text{gz}}(g, z)$ by applying the transformation in (7.1) to the bivariate polynomial L^A_{mz} given by (7.5) so that

$$L_{\text{gz}}(g, z) = -g(2\,z - 2\,z^2) - (1 - 2z). \tag{7.6}$$

Similarly, we can just as easily obtain the bivariate polynomial $L_{\text{rg}}(r, g)$ from L_{gz} by applying the transformation in (7.2). On doing so, we first obtain the intermediate function:

$$-g\left(2\left(r+\frac{1}{g}\right)-2\left(r+\frac{1}{g}\right)^2\right)-\left(1-2\left(r+\frac{1}{g}\right)\right)$$

which on simplification gives us the bivariate polynomial:

$$L_{\text{rg}}(r, g) = -1 + 2\,gr^2 + (2 - 2\,g)\,r. \tag{7.7}$$

The bivariate polynomial $L_{\text{sy}}(s, y)$ can be obtained from L_{mz} by applying the transformation in (7.3). On doing so, we first obtain the intermediate function:

$$(-s\,y)\left(2\,\frac{y+1}{sy}-2\,\left(\frac{y+1}{sy}\right)^2\right)-\left(1-2\frac{y+1}{sy}\right)$$

which on simplification gives us the bivariate polynomial:

$$L^A_{\text{sy}}(s, y) = (1 + 2\,y)\,s - 2 - 2\,y. \tag{7.8}$$

Table 3 tabulates all the bivariate polynomial encodings for the density in (7.4), the semi-circle law and the Marčenko-Pastur law.

Transform	Notation	Utility
Stieltjes transform	$m(z)$	Encodes spectral measure
Cauchy transform	$g(z)$	Same as above (different sign convention)
R transform	$r(g)$	Sum of random matrices
S transform	$s(y)$	Product of random matrices
Moment transform	$\mu(z)$	Ordinary moment generating function
Eta transform	$\eta(z)$	Wireless communications

Table 1: Transforms of the limiting spectral measure.

Procedure	MATLAB Code
	`function Luv = irreducLuv(Luv,u,v);`
Simplify and clear the denominator	`L = numden(simplify(expand(Luv)));`
	`L = Luv / maple('gcd',L,diff(L,u));`
Make square free	`L = simplify(expand(L));`
	`L = Luv / maple('gcd',L,diff(L,v));`
Simplify	`Luv = simplify(expand(L));`

Table 2: Making L_{uv} irreducible.

L	Bivariate Polynomials
L_{mz}	$m(2\,z-2\,z^2)-(1-2z)$
L_{gz}	$-g(2\,z-2\,z^2)-(1-2z)$
L_{rg}	$-1+2\,gr^2+(2-2\,g)\,r$
L_{sy}	$(1+2\,y)\,s-2-2\,y$
$L_{\mu\mathrm{z}}$	$(-2+2\,z)\,\mu+2-z$
$L_{\eta\mathrm{z}}$	$(2\,z+2)\,\eta-2-z$

(a) The atomic measure in (7.4).

L	Bivariate Polynomials
L_{mz}	$czm^2-(1-c-z)\,m+1$
L_{gz}	$czg^2+(1-c-z)\,g+1$
L_{rg}	$(cg-1)\,r+1$
L_{sy}	$(cy+1)\,s-1$
$L_{\mu\mathrm{z}}$	$\mu^2zc-(zc+1-z)\,\mu+1$
$L_{\eta\mathrm{z}}$	$\eta^2zc+(-zc+1-z)\,\eta-1$

(b) The Marčenko-Pastur law.

L	Bivariate polynomials
L_{mz}	$m^2+m\,z+1$
L_{gz}	$g^2-g\,z+1$
L_{rg}	$r-g$
L_{sy}	$s^2\,y-1$
$L_{\mu\mathrm{z}}$	$\mu^2z^2-\mu+1$
$L_{\eta\mathrm{z}}$	$z^2\eta^2-\eta+1$

(c) The semi-circle law.

Table 3: Bivariate polynomial encodings of some algebraic probability measures.

Label	Conversion	Transformation	MATLAB Code
I	$L_{\mathbf{mz}} \rightleftharpoons L_{\mathbf{gz}}$	$L_{\mathbf{mz}} = L_{\mathbf{gz}}(-m, z)$	function Lmz = Lgz2Lmz(Lgz) syms m g z Lmz = subs(Lgz,g,-m);
		$L_{\mathbf{gz}} = L_{\mathbf{mz}}(-g, z)$	function Lgz = Lmz2Lgz(Lmz) syms m g z Lgz = subs(Lmz,m,-g);
II	$L_{\mathbf{gz}} \rightleftharpoons L_{\mathbf{rg}}$	$L_{\mathbf{gz}} = L_{\mathbf{rg}}(z - \frac{1}{g}, z)$	function Lgz = Lrg2Lgz(Lrg) syms r g z Lgz = subs(Lrg,r,z-1/g); Lgz = irreducLuv(Lgz,g,z);
		$L_{\mathbf{rg}} = L_{\mathbf{gz}}(g, r + \frac{1}{g})$	function Lrg = Lgz2Lrg(Lgz) syms r g z Lrg = subs(Lgz,g,r+1/g); Lrg = irreducLuv(Lrg,r,g);
III	$L_{\mathbf{mz}} \rightleftharpoons L_{\mathbf{rg}}$	$L_{\mathbf{mz}} \rightleftharpoons L_{\mathbf{gz}} \rightleftharpoons L_{\mathbf{rg}}$	function Lmz = Lrg2Lmz(Lrg) syms m z r g Lgz = Lrg2Lgz(Lrg); Lmz = Lgz2Lmz(Lgz);
			function Lrg = Lmz2Lrg(Lmz) syms m z r g Lgz = Lmz2Lgz(Lmz); Lrg = Lgz2Lrg(Lgz);
IV	$L_{\mathbf{mz}} \rightleftharpoons L_{\mathbf{sy}}$	$L_{\mathbf{mz}} = L_{\mathbf{sy}}(\frac{m}{z\,m+1}, -z\,m - 1)$	function Lmz = Lsy2Lmz(Lsy) syms m z s y Lmz = subs(Lsy,s,m/(z*m+1)); Lmz = subs(Lmz,y,-z*m-1); Lmz = irreducLuv(Lmz,m,z);
		$L_{\mathbf{sy}} = L_{\mathbf{mz}}(-y\,s, \frac{y+1}{s\,y})$	function Lsy = Lmz2Lsy(Lmz) syms m z s y Lsy = subs(Lmz,m,-y*s); Lsy = subs(Lsy,z,(y+1)/y/s); Lsy = irreducLuv(Lsy,s,y);
V	$L_{\mathbf{mz}} \rightleftharpoons L_{\mu\mathbf{z}}$	$L_{\mathbf{mz}} = L_{\mu\mathbf{z}}(-m\,z, \frac{1}{z})$	function Lmz = Lmyuz2Lmz(Lmyuz) syms m myu z Lmz = subs(Lmyuz,z,1/z); Lmz = subs(Lmz,myu,-m*z); Lmz = irreducLuv(Lmz,m,z);
		$L_{\mu\mathbf{z}} = L_{\mathbf{mz}}(-\mu\,z, \frac{1}{z})$	function Lmyuz = Lmz2Lmyuz(Lmz) syms m myu z Lmyuz = subs(Lmz,z,1/z); Lmyuz = subs(Lmyuz,m,-myu*z); Lmyuz = irreducLuv(Lmyuz,myu,z);
VI	$L_{\mathbf{mz}} \rightleftharpoons L_{\eta\mathbf{z}}$	$L_{\mathbf{mz}} = L_{\eta\mathbf{z}}(-z\,m, -\frac{1}{z})$	function Lmz = Letaz2Lmz(Letaz) syms m eta z Lmz = subs(Letaz,z,-1/z); Lmz = subs(Lmz,eta,-z*m); Lmz = irreducLuv(Lmz,m,z);
		$L_{\eta\mathbf{z}} = L_{\mathbf{mz}}(z\,\eta, -\frac{1}{z})$	function Letaz = Lmz2Letaz(Lmz) syms m eta z Letaz = subs(Lmz,z,-1/z); Letaz = subs(Letaz,m,z*eta); Letaz = irreducLuv(Letaz,eta,z);

Table 4: Transformations between the different bivariate polynomials. As a guide to MATLAB notation, the command `syms` declares a variable to be symbolic while the command `subs` symbolically substitutes every occurrence of the second argument in the first argument with the third argument. Thus, for example, the command `y=subs(x-a,a,10)` will yield the output `y=x-10` if we have previously declared x and a to be symbolic using the command `syms x a`.

8. Algebraic operations on algebraic functions

Algebraic functions are closed under addition and multiplication. Hence we can add (or multiply) two algebraic functions and obtain another algebraic function. We show, using purely matrix theoretic arguments, how to obtain the polynomial whose zeros are the sum (or product) of two algebraic functions without ever actually computing the roots. In Section 9.1 we interpret this computation using the concept of resultants [21] from elimination theory.

Definition 8.1 (Companion Matrix). *The companion matrix* $\mathbf{C}_{a(x)}$ *to a monic polynomial*

$$a(x) \equiv p_0 + p_1\, x + \ldots + a_{n-1}\, x^{n-1} + x^n$$

is the $n \times n$ *square matrix*

$$\mathbf{C}_{a(x)} = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & -a_0 \\ 1 & \cdots & \cdots & \cdots & -a_1 \\ 0 & \ddots & & & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & \ldots & 1 & -a_{n-1} \end{bmatrix}$$

with ones on the sub-diagonal and the last column given by the negative coefficients of $a(x)$.

Remark 8.2. *The eigenvalues of the companion matrix are equal to the roots of* $a(x)$*. This is intimately related to the observation that the characteristic polynomial of the companion matrix equals* $a(x)$*, i.e.,*

$$a(x) = \det(x\, \mathbf{I}_n - \mathbf{C}_{a(x)}).$$

Consider the bivariate polynomial L_{uv} as in (3.1). By treating it as a polynomial in u whose coefficients are polynomials in v:

$$L_{\mathrm{uv}}(u, v) \equiv \sum_{j=0}^{\mathrm{D_u}} l_j(v)\, u^j, \tag{8.1}$$

we can create a companion matrix $\mathbf{C}_{\mathrm{uv}}^{\mathrm{u}}$ whose characteristic polynomial as a function of u is the bivariate polynomial L_{uv}. The companion matrix $\mathbf{C}_{\mathrm{uv}}^{\mathrm{u}}$ is the $\mathrm{D_u} \times \mathrm{D_u}$ matrix in Table 5.

$\mathbf{C}_{\mathrm{uv}}^{\mathrm{u}}$	MATLAB code

$\begin{bmatrix} 0 & \ldots & \ldots & \ldots & -l_0(v)/l_{\mathrm{D_u}}(v) \\ 1 & \cdots & \cdots & \cdots & -l_1(v)/l_{\mathrm{D_u}}(v) \\ 0 & \ddots & & & -l_2(v)/l_{\mathrm{D_u}}(v) \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & \ldots & 1 & -l_{\mathrm{D_u}-1}(v)/l_{\mathrm{D_u}}(v) \end{bmatrix}$	<pre>function Cu = Luv2Cu(Luv,u)
	Du = double(maple('degree',Luv,u));
	LDu = maple('coeff',Luv,u,Du);
	Cu = sym(zeros(Du))+diag(ones(Du-1,1),-1));
	
	for Di = 0:Du-1
	 Cu(Di+1,Du) = -maple('coeff',Lt,u,Di)/LDu;
	end</pre>

Table 5: The companion matrix $\mathbf{C}_{\mathrm{uv}}^{\mathrm{u}}$, with respect to u, of the bivariate polynomial L_{uv} given by (8.1).

Remark 8.3. *Analogous to the univariate case, the characteristic polynomial* $\det(u\,\mathbf{I} - \mathbf{C}_{\mathrm{uv}}^{\mathrm{u}}) = L_{\mathrm{uv}}(u, v)$*, where the equality is understood to be with respect to the equivalence class of* L_{uv}*. The "eigenvalues" (eigenfunctions to be exact) of* $\mathbf{C}_{\mathrm{uv}}$ *are the roots of the algebraic equation* $L_{\mathrm{uv}}(u, v) = 0$*; specifically we obtain the algebraic function* $u(v)$*.*

Definition 8.4 (Kronecker product). *If $\mathbf{A}_m$ (with entries a_{ij}) is an $m \times m$ matrix and $\mathbf{B}_n$ is a $n \times n$ matrix then the Kronecker or tensor product of $\mathbf{A}_m$ and $\mathbf{B}_n$, denoted by $\mathbf{A}_m \otimes \mathbf{B}_n$, is the $mn \times mn$ matrix defined as:*

$$\mathbf{A}_m \otimes \mathbf{B}_n = \begin{bmatrix} a_{11}\mathbf{B}_n & \dots & a_{1n}\mathbf{B}_n \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B}_n & \dots & a_{mn}\mathbf{B}_n \end{bmatrix}$$

Theorem 8.5. *If α_i and β_j are the eigenvalues of $\mathbf{A}_m$ and $\mathbf{B}_n$ respectively, then*

1. *$\alpha_i + \beta_j$ is an eigenvalue of $(\mathbf{A}_m \otimes \mathbf{I}_n) + (\mathbf{I}_m \otimes \mathbf{B}_n)$,*
2. *$\alpha_i\,\beta_j$ is an eigenvalue of $\mathbf{A}_m \otimes \mathbf{B}_n$,*

for $i = 1, \dots, m$, $j = 1, \dots, n$.

Corollary 8.1. *Let $u_1(v)$ be a root of the algebraic function L^1_{uv}, or equivalently an eigenvalue of the $D^1_{\mathrm{u}} \times D^1_{\mathrm{u}}$ companion matrix $\mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}}$. Let $u_2(v)$ be a root of the algebraic function L^2_{uv}, or equivalently an eigenvalue of the $D^2_{\mathrm{u}} \times D^2_{\mathrm{u}}$ companion matrix $\mathbf{C}^{\mathrm{u}_2}_{\mathrm{uv}}$. Then*

1. *$u_3(v) = u_1(v) + u_2(v)$ is an eigenvalue of the matrix $\mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}} = \left(\mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}} \otimes \mathbf{I}_{D^2_{\mathrm{u}}}\right) + \left(\mathbf{I}_{D^1_{\mathrm{u}}} \otimes \mathbf{C}^{\mathrm{u}_2}_{\mathrm{uv}}\right)$,*
2. *$u_3(v) = u_1(v) u_2(v)$ is an eigenvalue of the matrix $\mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}} = \mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}} \otimes \mathbf{C}^{\mathrm{u}_2}_{\mathrm{uv}}$.*

Equivalently $u_3(v)$ is the root of the algebraic function $L^3_{\mathrm{uv}} = \det(u\,\mathbf{I} - \mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}})$.

We represent the binary addition and multiplication operators on the space of algebraic functions by the symbols $\boxplus_{\mathrm{u}}$ and $\boxtimes_{\mathrm{u}}$ respectively. Addition and multiplication are defined as in Table 6. Note that the subscript 'u' in $\boxplus_{\mathrm{u}}$ and $\boxtimes_{\mathrm{u}}$ provides us with an indispensable convention of which dummy variable we are using. Table 7 illustrates the $\boxplus$ and $\boxtimes$ operations on a pair of bivariate polynomials. No human need try and replicate these by hand. The $\mathrm{D_u} + 1 \times \mathrm{D_v} + 1$ matrix $\mathbf{T}_{uv}$ lists only the coefficients c_{ij} for the term $u^i\,v^j$ in the polynomial $L_{\mathrm{uv}}(u, v)$. Note that the indexing for i and j starts with zero.

<table>
<tr><th>Operation: $L^1_{\mathrm{uv}}, L^2_{\mathrm{uv}} \longmapsto L^3_{\mathrm{uv}}$</th><th>MATLAB Code</th></tr>
<tr><td>$L^3_{\mathrm{uv}} = L^1_{\mathrm{uv}} \boxplus_{\mathrm{u}} L^2_{\mathrm{uv}} \equiv \det(u\,\mathbf{I} - \mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}})$, where
$\mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}} = \begin{cases} 2\,\mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}} & \text{if } L^1_{\mathrm{uv}} = L^2_{\mathrm{uv}}, \\ (\mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}} \otimes \mathbf{I}_{D^2_{\mathrm{u}}}) + (\mathbf{I}_{D^1_{\mathrm{u}}} \otimes \mathbf{C}^{\mathrm{u}_2}_{\mathrm{uv}}) & \text{otherwise.} \end{cases}$</td><td>function Luv3 = L1plusL2(Luv1,Luv2,u)
Cu1 = Luv2Cu(Luv1,u);
if (Luv1 == Luv2)
  Cu3 = 2*Cu1;
else
  Cu2 = Luv2Cu(Luv2,u);
  Cu3 = kron(Cu1,eye(length(Cu2))) + ..
      +kron(eye(length(Cu1)),Cu2);
end
Luv3 = det(u*eye(length(Cu3))-Cu3);</td></tr>
<tr><td>$L^3_{\mathrm{uv}} = L^1_{\mathrm{uv}} \boxtimes_{\mathrm{u}} L^2_{\mathrm{uv}} \equiv \det(u\,\mathbf{I} - \mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}})$, where
$\mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}} = \begin{cases} \mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}} = (\mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}})^2 & \text{if } L^1_{\mathrm{uv}} = L^2_{\mathrm{uv}}, \\ \mathbf{C}^{\mathrm{u}_3}_{\mathrm{uv}} = \mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}} \otimes \mathbf{C}^{\mathrm{u}_2}_{\mathrm{uv}} & \text{otherwise.} \end{cases}$</td><td>function Luv3 = L1timesL2(Luv1,Luv2,u)
Cu1 = Luv2Cu(Luv1,u);
if (Luv1 == Luv2)
  Cu3 = Cu2̂;
else
  Cu2 = Luv2Cu(Luv2,u);
  Cu3 = kron(Cu1,Cu2);
end
Luv3 = det(u*eye(length(Cu3))-Cu3);</td></tr>
</table>

Table 6: Formal and computational description of the $\boxplus_{\mathrm{u}}$ and $\boxtimes_{\mathrm{u}}$ operators acting on the bivariate polynomials $L^1_{\mathrm{uv}}(u, v)$ and $L^2_{\mathrm{uv}}(u, v)$ where $\mathbf{C}^{\mathrm{u}_1}_{\mathrm{uv}}$ and $\mathbf{C}^{\mathrm{u}_2}_{\mathrm{uv}}$ are their corresponding companion matrices constructed as in Table 5 and $\otimes$ is the matrix Kronecker product.

L_{uv}	$\mathbf{T}_{\mathrm{uv}}$	$\mathbf{C}_{\mathrm{uv}}^{\mathrm{u}}$	$\mathbf{C}_{\mathrm{uv}}^{\mathrm{v}}$
$L_{\mathrm{uv}}^{1} \equiv u^{2}v + u\,(1-v) + v^{2}$	$\begin{array}{c c} & \begin{array}{ccc} 1 & v & v^2 \end{array} \\ \begin{array}{c} 1 \\ u \\ u^2 \end{array} & \left[\begin{array}{ccc} \cdot & \cdot & 1 \\ 1 & -1 & \cdot \\ \cdot & 1 & \cdot \end{array}\right] \end{array}$	$\begin{bmatrix} 0 & -v \\ 1 & \dfrac{-1+v}{v} \end{bmatrix}$	$\begin{bmatrix} 0 & -u \\ 1 & -u^2+u \end{bmatrix}$
$L_{\mathrm{uv}}^{2} \equiv u^{2}\left(v^{2} - 3\,v + 1\right) + u\,(1+v) + v^{2}$	$\begin{array}{c c} & \begin{array}{ccc} 1 & v & v^2 \end{array} \\ \begin{array}{c} 1 \\ u \\ u^2 \end{array} & \left[\begin{array}{ccc} \cdot & \cdot & 1 \\ 1 & 1 & \cdot \\ 1 & -3 & 1 \end{array}\right] \end{array}$	$\begin{bmatrix} 0 & \dfrac{-v^2}{v^2-3\,v+1} \\ 1 & \dfrac{-1-v}{v^2-3\,v+1} \end{bmatrix}$	$\begin{bmatrix} 0 & \dfrac{-u^2-u}{u^2+1} \\ 1 & \dfrac{3\,u^2-u}{u^2+1} \end{bmatrix}$

$L_{\mathrm{uv}}^{1} \boxplus_{\mathrm{u}} L_{\mathrm{uv}}^{2}$	$\begin{array}{c c} & \begin{array}{ccccccccc} 1 & v & v^2 & v^3 & v^4 & v^5 & v^6 & v^7 & v^8 \end{array} \\ \begin{array}{c} 1 \\ u \\ u^2 \\ u^3 \\ u^4 \end{array} & \left[\begin{array}{ccccccccc} \cdot & \cdot & 2 & -6 & 11 & -10 & 18 & -8 & 1 \\ 2 & \cdot & 2 & -8 & 4 & \cdot & \cdot & \cdot & \cdot \\ 5 & \cdot & 1 & -4 & 2 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \end{array}$
$L_{\mathrm{uv}}^{1} \boxtimes_{\mathrm{u}} L_{\mathrm{uv}}^{2}$	$\begin{array}{c c} & \begin{array}{ccccccccccccccc} 1 & v & v^2 & v^3 & v^4 & v^5 & v^6 & v^7 & v^8 & v^9 & v^{10} & v^{11} & v^{12} & v^{13} & v^{14} \end{array} \\ \begin{array}{c} 1 \\ u \\ u^2 \\ u^3 \\ u^4 \end{array} & \left[\begin{array}{ccccccccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -6 & 11 & -6 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 3 & \cdot & -3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -4 & 10 & -6 & 7 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \end{array}$

$L_{\mathrm{uv}}^{1} \boxplus_{\mathrm{v}} L_{\mathrm{uv}}^{2}$	$L_{\mathrm{uv}}^{2} \boxtimes_{\mathrm{v}} L_{\mathrm{uv}}^{2}$
$\begin{array}{c c} & \begin{array}{ccccc} 1 & v & v^2 & v^3 & v^4 \end{array} \\ \begin{array}{c} 1 \\ u \\ u^2 \\ u^3 \\ u^4 \\ u^5 \\ u^6 \\ u^7 \\ u^8 \end{array} & \left[\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 4 & \cdot & \cdot \\ \cdot & \cdot & 1 & -4 & \cdot \\ \cdot & -8 & 6 & \cdot & \cdot \\ 1 & -2 & 3 & \cdot & \cdot \\ 8 & -12 & \cdot & \cdot & \cdot \\ 3 & 2 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \end{array}\right] \end{array}$	$\begin{array}{c c} & \begin{array}{ccccc} 1 & v & v^2 & v^3 & v^4 \end{array} \\ \begin{array}{c} 1 \\ u \\ u^2 \\ u^3 \\ u^4 \\ u^5 \\ u^6 \\ u^7 \\ u^8 \\ u^9 \\ u^{10} \end{array} & \left[\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2 & 1 & \cdot \\ \cdot & \cdot & \cdot & -4 & \cdot \\ 1 & 1 & -9 & 3 & \cdot \\ 2 & -3 & 7 & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & -1 & \cdot & \cdot \\ 3 & -1 & 1 & \cdot & \cdot \\ 2 & 3 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{array}\right] \end{array}$

Table 7: Examples of $\boxplus$ and $\boxtimes$ operations on a pair of bivariate polynomials, L_{uv}^{1} and L_{uv}^{2}.

$\mathbf{B}$	Operation	$L^{B}_{\mathbf{mz}}(m,z)$	MATLAB code
		Deterministic Transformations	

$\frac{p\,\mathbf{A}+q\,\mathbf{I}}{r\,\mathbf{A}+s\,\mathbf{I}}$	''Mobius''	$L^{A}_{\mathbf{mz}}\left(\frac{m-\beta_z\, r}{\beta_z\, s-\beta_z\, r\,\alpha_z}, -\alpha_z\right)$, where $\alpha_z = (q - s\,z)/(p - r\,z)$, and $\beta_z = 1/(p - r\,z)$.	function LmzB = mobiusA(LmzA,p,q,r,s)
			syms m z
			alpha = ((q-s*z)/(p-r*z);beta=1/(p-r*z);
			temp_pol = subs(LmzA,z,-alpha);
			temp_pol = subs(temp_pol,m,((m/beta)-r)/(s-r*alpha));
			LmzB = irreducLuv(temp_pol,m,z);

$\mathbf{A}^{-1}$	''Invert''	$L^{A}_{\mathbf{mz}}\left(-z - z^2\,m, \frac{1}{z}\right)$	function LmzB = invA(LmzA) LmzB = mobiusA(LmzA,0,1,1,0);
$\mathbf{A}+\alpha\,\mathbf{I}$	''Translate''	$L^{A}_{\mathbf{mz}}(m, z-\alpha)$	function LmzB = shiftA(LmzA,alpha) LmzB = mobiusA(LmzA,1,alpha,0,1);
$\alpha\,\mathbf{A}$	''Scale''	$L^{A}_{\mathbf{mz}}\left(\alpha\, m, \frac{z}{\alpha}\right)$	function LmzB = scaleA(LmzA) LmzB = mobiusA(LmzA,alpha,0,0,1);

$\begin{bmatrix}\mathbf{A} & 0\\ 0 & \alpha\mathbf{I}\end{bmatrix}$	''Projection/ Transpose''	$\frac{\text{Size of }\mathbf{A}}{\text{Size of }\mathbf{B}} \to c > 1$
		$L^{A}_{\mathbf{mz}}\left((1-\frac{1}{c})\frac{1}{\alpha - z} + \frac{m}{c}, z\right)$
			function LmzB = projectA(LmzA,c,alpha)
			syms m z
			mb = (1-(1/c))*(1/(alpha-z))+m/c;
			temp_pol = subs(LmzA,m,mb);
			LmzB = irreducLuv(temp_pol,m,z);

$\mathbf{A} = \begin{bmatrix}\mathbf{B} & 0\\ 0 & \alpha\mathbf{I}\end{bmatrix}$	''Augmentation''	$\frac{\text{Size of }\mathbf{A}}{\text{Size of }\mathbf{B}} \to c < 1$	function LmzB = augmentA(LmzA,c,alpha)
			syms m z
			mb = (1-(1/c))*(1/(alpha-z))+m/c;
			temp_pol = subs(LmzA,m,mb);
			LmzB = irreducLuv(temp_pol,m,z);

		Stochastic Transformations	

$\mathbf{A}+\mathbf{G}'\,\mathbf{T}\,\mathbf{G}$	''Add Atomic Wishart''	$L^{A}_{\mathbf{mz}}(m, z-\alpha_m)$, where $\alpha_m = c\sum_{i=1}^{d}\frac{p_i\,\lambda_i}{1+\lambda_i\, m}$, with $\sum_i p_i = 1$.	function LmzB = AplusWish(LmzA,c,p,lambda)
			syms m z
			alpha = z-c*sum(p.*(lambda./(1+lambda*m)));
			temp_pol = subs(LmzA,z,z-alpha);
			LmzB = irreducLuv(temp_pol,m,z);

$\mathbf{A}\times\mathbf{W}(c)$	''Multiply Wishart''	$L^{A}_{\mathbf{mz}}\left(\alpha_{m,z}\, m, \frac{z}{\alpha_{m,z}}\right)$, where $\alpha_{m,z} = (1 - c - c\,z\,m)$.	function LmzB = AtimesWish(LmzA,c)
			syms m z z1
			alpha = (1-c-c*z1*m); temp_pol = subs(LmzA,m,m*alpha);
			temp_pol = subs(temp_pol,z,z1/alpha);
			temp_pol = subs(temp_pol,z1,z); % Replace dummy variable
			LmzB = irreducLuv(temp_pol,m,z);

$(\mathbf{A}^{1/2}+\sqrt{s}\,\mathbf{G})$ $\times$ $(\mathbf{A}^{1/2}+\sqrt{s}\,\mathbf{G})'$	''Grammian''	$L^{A}_{\mathbf{mz}}\left(\frac{m}{\alpha_m}, \alpha_m^2\, z + \alpha_m\, s(c-1)\right)$, where $\alpha_m = 1 + s\,c\,m$.	function LmzB = AgramG(LmzA,c,s)
			syms m z
			alpha = (1+s*c*m); beta = alpha*(z*alpha+s*(c-1));
			temp_pol = subs(subs(LmzA,m,m/alpha),z,beta);
			LmzB = irreducLuv(temp_pol,m,z);

Table 8: Operational laws on the bivariate polynomial encodings (and their computational realization in MATLAB) corresponding to a class of deterministic and stochastic transformations. The Gaussian-like random matrix $\mathbf{G}$ is an $N \times L$, the Wishart-like matrix $\mathbf{W}(c) = \mathbf{G}\,\mathbf{G}'$ where $N/L \to c > 0$ as $N, L \to \infty$, and the matrix $\mathbf{T}$ is a diagonal atomic random matrix.

Operational Law	MATLAB Code

$\begin{array}{ccc} & L^A_{\mathrm{mz}} & \\ \swarrow & & \searrow \\ L^A_{\mathrm{mz}}(2m\sqrt{z},\sqrt{z}) & & L^A_{\mathrm{mz}}(-2m\sqrt{z},-\sqrt{z}) \\ \searrow & & \swarrow \\ & \boxplus_{\mathrm{m}} & \\ & \downarrow & \\ & L^B_{\mathrm{mz}} & \end{array}$	`function LmzB = squareA(LmzA)`
	`syms m z`
	
	`Lmz1 = subs(LmzA,z,sqrt(z));`
	`Lmz1 = subs(Lmz1,m,2*m*sqrt(z));`
	`Lmz2 = subs(LmzA,z,-sqrt(z));`
	`Lmz2 = subs(Lmz2,m,-2*m*sqrt(z));`
	
	`LmzB = L1plusL2(Lmz1,Lmz2,m);`
	`LmzB = irreducLuv(LmzB,m,z);`

(a) $L^A_{\mathrm{mz}} \longmapsto L^B_{\mathrm{mz}}$ for $\mathbf{A} \longmapsto \mathbf{B} = \mathbf{A}^2$.

Operational Law	MATLAB Code

$\begin{array}{ccc} L^A_{\mathrm{mz}} & & L^B_{\mathrm{mz}} \\ \downarrow & & \downarrow \\ L^A_{\mathrm{mz}}(\frac{m}{c},z) & & L^B_{\mathrm{mz}}(\frac{m}{1-c},z) \\ \searrow & & \swarrow \\ & \boxplus_{\mathrm{m}} & \\ & \downarrow & \\ & L^C_{\mathrm{mz}} & \end{array}$	`function LmzC = AblockB(LmzA,LmzB,c)`
	`syms m z mu`
	
	`LmzA1 = subs(LmzA,m,m/c);`
	`LmzB1 = subs(LmzB,m,m/(1-c));`
	
	`LmzC = L1plusL2(LmzA1,LmzB1,m);`
	`LmzC = irreducLuv(LmzC,m,z);`

(b) $L^A_{\mathrm{mz}}, L^A_{\mathrm{mz}} \longmapsto L^C_{\mathrm{mz}}$ for $\mathbf{A}, \mathbf{B} \longmapsto \mathbf{C} = \mathrm{diag}(\mathbf{A}, \mathbf{B})$ where Size of $\mathbf{A}$/ Size of $\mathbf{C} \to c$.

Operational Law	MATLAB Code

$\begin{array}{ccc} L^A_{\mathrm{mz}} & & L^B_{\mathrm{mz}} \\ \downarrow & & \downarrow \\ L^A_{\mathrm{rg}} & & L^B_{\mathrm{rg}} \\ \searrow & & \swarrow \\ & \boxplus_{\mathrm{r}} & \\ & \downarrow & \\ & L^C_{\mathrm{rg}} & \\ & \downarrow & \\ & L^C_{\mathrm{mz}} & \end{array}$	`function LmzC = AplusB(LmzA,LmzB)`
	`syms m z r g`
	
	`LrgA = Lmz2Lrg(LmzA);`
	`LrgB = Lmz2Lrg(LmzB);`
	
	`LrgC = L1plusL2(LrgA,LrgB,r);`
	
	`LmzC = Lrg2Lmz(LrgC);`

(c) $L^A_{\mathrm{mz}}, L^B_{\mathrm{mz}} \longmapsto L^C_{\mathrm{mz}}$ for $\mathbf{A}, \mathbf{B} \longmapsto \mathbf{C} = \mathbf{A} + \mathbf{Q}\mathbf{B}\mathbf{Q}'$.

Operational Law	MATLAB Code

$\begin{array}{ccc} L^A_{\mathrm{mz}} & & L^B_{\mathrm{mz}} \\ \downarrow & & \downarrow \\ L^A_{\mathrm{sy}} & & L^B_{\mathrm{sy}} \\ \searrow & & \swarrow \\ & \boxtimes_{\mathrm{s}} & \\ & \downarrow & \\ & L^C_{\mathrm{sy}} & \\ & \downarrow & \\ & L^C_{\mathrm{mz}} & \end{array}$	`function LmzC = AtimesB(LmzA,LmzB)`
	`syms m z s y`
	
	`LsyA = Lmz2Lsy(LmzA);`
	`LsyB = Lmz2Lsy(LmzB);`
	
	`LsyC = L1timesL2(LsyA,LsyB,s);`
	
	`LmzC = Lsy2Lmz(LsyC);`

(d) $L^A_{\mathrm{mz}}, L^B_{\mathrm{mz}} \longmapsto L^C_{\mathrm{mz}}$ for $\mathbf{A}, \mathbf{B} \longmapsto \mathbf{C} = \mathbf{A} \times \mathbf{Q}\mathbf{B}\mathbf{Q}'$.

Table 9: Operational laws on the bivariate polynomial encodings for some canonical random matrix transformations. The operations $\boxplus_{\mathrm{u}}$ and $\boxtimes_{\mathrm{u}}$ are defined in Table 6.

9. Operational laws on the bivariate polynomials

Tables 8 and 9 summarize the random matrix transformations and the corresponding operational laws on the bivariate polynomials. This demonstrates that the algebraicity of the limiting measures is preserved when an algebraic random matrix is canonically transformed. Section 10. details the proofs. We now highlight the connection between the $\boxplus$ and $\boxtimes$ operations, resultants and free convolution of algebraic probability measures.

9.1 The resultant connection

Definition 9.1 (Resultant). *Given a polynomial*

$$a(x) \equiv a_0 + a_1\, x + \ldots + a_{n-1}\, x^{n-1} + a_n x^n$$

of degree n with roots α_i, for $i = 1, \ldots, n$ and a polynomial

$$b(x) \equiv b_0 + b_1\, x + \ldots + b_{m-1}\, x^{m-1} + b_m x^m$$

of degree m with roots β_j, for $j = 1, \ldots, m$, the resultant is defined by:

$$\mathrm{Res}_{\,\mathrm{x}}\,(a(x)\,,\, b(x)) = \alpha_n^m\, \beta_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\beta_j - \alpha_i).$$

The resultant is given by the determinant of the Sylvester matrix formed using just the coefficients of the associated polynomials. For our purpose, the $\boxplus_{\mathrm{u}}$ and $\boxtimes_{\mathrm{u}}$ operations can be expressed in terms of resultants. Suppose we are given two bivariate polynomials L^A_{uv} and L^B_{uv}. By using the definition of the resultant and treating the bivariate polynomials as polynomials in u whose coefficients are polynomials in v, it can be shown that:

$$\boxed{L^C_{\mathrm{uv}}(t,v) = L^A_{\mathrm{uv}} \boxplus_{\mathrm{u}} L^B_{\mathrm{uv}} \equiv \mathrm{Res}_{\,\mathrm{u}}\left(L^A_{\mathrm{uv}}(t-u,v)\,,\, L^B_{\mathrm{uv}}(u,v)\right),} \tag{9.1}$$

and

$$\boxed{L^C_{\mathrm{uv}}(t,v) = L^A_{\mathrm{uv}} \boxtimes_{\mathrm{u}} L^B_{\mathrm{uv}} \equiv \mathrm{Res}_{\,\mathrm{u}}\left(u^{D^A_{\mathrm{u}}} L^A_{\mathrm{uv}}(t/u,v)\,,\, L^B_{\mathrm{uv}}(u,v)\right),} \tag{9.2}$$

where D^A_{u} is the degree of L^A_{uv} with respect to u. Fast algorithms for computing the resultant are available through Maple's `resultant` command. In Maple, the computation $L^C_{\mathrm{uv}} = L^A_{\mathrm{uv}} \boxplus_{\mathrm{u}} L^B_{\mathrm{uv}}$ may be performed as:

```
LuvC = subs(t=u,resultant(subs(u=t-u,LuvA),LuvB,u));
```

The computation $L^C_{\mathrm{uv}} = L^A_{\mathrm{uv}} \boxtimes_{\mathrm{u}} L^B_{\mathrm{uv}}$ can be performed via the sequence of commands:

```
DuA = degree(LuvA,u);
LuvC =  subs(t=u,resultant(simplify(u^DuA*subs(u=t/u,LuvA)),LuvB,u));
```

There is an explicit connection between resultants and the free convolution of algebraic probability measures.

9.2 Connection with free convolution

First, a quick word on "freeness" and "free probability". There is a whole mathematics of free probability emerging as a counterpart to classical probability. Some good references are [9, 26]. These references and even the name "free probability" are worthy of some introduction.

We begin with a viewpoint on classical probability. If we are given probability densities f_X and f_Y for random variables X and Y respectively, and if we know that X and Y are independent, we can compute the moments of $X + Y$, and XY, for example, from the moments of X and Y.

Free additive convolution	$f_{A+B} = f_A \boxplus f_B$	$L_{\mathrm{rg}}^{A+B} = L_{\mathrm{rg}}^{A} \boxplus_{\mathrm{r}} L_{\mathrm{rg}}^{B}$
Free multiplicative convolution	$f_{A\times B} = f_A \boxtimes f_B$	$L_{\mathrm{sy}}^{A\times B} = L_{\mathrm{sy}}^{A} \boxtimes_{\mathrm{s}} L_{\mathrm{sy}}^{B}$

Table 10: Free convolution of two algebraic probability measures can be expressed as the resultant of a pair of bivariate polynomials.

Our viewpoint on free probability is similar. Given two random matrices, $\mathbf{A}_N$ and $\mathbf{B}_N$ with limiting spectral measures f_A and f_B, we would like to compute the limiting spectral measure for $\mathbf{A}_N + \mathbf{B}_N$ and $\mathbf{A}_N\mathbf{B}_N$ in terms of the moments of f_A and f_B.

Of course, (suppressing the subscript N for brevity) $\mathbf{A}$ and $\mathbf{B}$ do not commute so we are in the realm of non-commutative algebra. Since all possible products of $\mathbf{A}$ and $\mathbf{B}$ are allowed we have the "free" product, i.e., all words in $\mathbf{A}$ and $\mathbf{B}$ are allowed. (We recall that this is precisely the definition of the free product in algebra.) The theory of free probability allows us to compute the moments of these products in the limit of large matrices so long as at least one of $\mathbf{A}$ or $\mathbf{B}$ has what amounts to eigenvectors that essentially are uniformly distributed with Haar measure. Speicher's work [19] places these moment computations in an elegant combinatorial context.

Consequently, the limiting spectral measure of the sum of two "free" random matrices is obtained by the free additive convolution of the individual limiting spectral measures. Similarly, where the product has real eigenvalues, the spectral measure of the product of two "free" random matrices is the free multiplicative convolution of the individual spectral measures. It turns out that the free convolution of algebraic probability measures is also an algebraic probability measure. The convolution can be expressed as a resultant of the appropriate bivariate polynomial as summarized in Table 9.2. For additional examples see [17].

10. Proofs

We now prove the operational laws on the bivariate polynomials and the associated mode of convergence. The continuous mapping theorem, stated below, follows from well-known facts about the convergence of probability measures [3].

Theorem 10.1 (Continuous mapping theorem). *Let $\mathbf{A}_N \to A$. Let f_A and $\mathcal{S}_A^{\delta}$ denote the corresponding limiting spectral measure and the atomic component of the support, respectively. Consider the mapping $y = h(x)$ continuous everywhere on the real line except on the set of its discontinuities denoted by $\mathcal{D}_h$. If $D_h \cap \mathcal{S}_A^{\delta} = \emptyset$ then $\mathbf{B}_N = h(\mathbf{A}_N) \to B$. The associated non-random d.f., F^B is given by $F^B(y) = F^A\left(h^{\langle -1\rangle}(y)\right)$. The associated probability measure is simply its distributional derivative.*

10.1 $\mathbf{A}_N \longmapsto \mathbf{B}_N = (p\,\mathbf{A}_N + q\,\mathbf{I}_N)/(r\,\mathbf{A}_N + s\,\mathbf{I}_N)$

Here we have $h(x) = (p\,x + r)/(q\,x + s)$ which is continuous everywhere except at $x = -s/r$ for s and r not simultaneously zero. From Theorem 10.1, unless $f_A(x)$ has an atomic component at $-s/r$, $\mathbf{B}_N \to B$. The Stieltjes transform of f_B can be expressed as:

$$m_B(z) = E_Y\left[\frac{1}{y-z}\right] = E_X\left[\frac{r\,x+s}{p\,x+q-z(r\,x+s)}\right]. \tag{10.1}$$

Equation (10.1) can be rewritten as

$$m_B(z) = \int \frac{r\,x + s}{(p - r\,z)x + (q - s\,z)} f_A(x)dx = \frac{1}{p - r\,z} \int \frac{r\,x + s}{x + \frac{q - s\,z}{p - r\,z}} f_A(x)dx. \tag{10.2}$$

With some algebraic manipulations, we can rewrite (10.2) as

$$\begin{aligned} m_B(z) &= \beta_z \int \frac{r\,x + s}{x + \alpha_z} f_A(x)dx = \beta_z \left(r \int \frac{x}{x + \alpha_z} f_A(x)dx + s \int \frac{1}{x + \alpha_z} f_A(x)dx \right) \\ &= \beta_z \left(r \int f_A(x)dx - r\,\alpha_z \int \frac{1}{x + \alpha_z} f_A(x)dx + s \int \frac{1}{x + \alpha_z} f_A(x)dx \right). \end{aligned} \tag{10.3}$$

where $\beta_z = 1/(p - r\,z)$ and $\alpha_z = (q - s\,z)/(p - r\,z)$. Using the definition of the Stieltjes transform and the identity $\int f_A(x)dx = 1$, we can express $m_B(z)$ in (10.3) in terms of $m_A(z)$ as simply

$$m_B(z) = \beta_z\,r + (\beta_z\,s - \beta\,r\,\alpha_z)\,m_A(-\alpha_z). \tag{10.4}$$

Equation (10.4) can, equivalently, also be rewritten as

$$m_A(-\alpha_z) = \frac{m_B(z) - \beta_z\,r}{\beta_z\,s - \beta_z\,r\,\alpha_z}. \tag{10.5}$$

Equation (10.5) can be expressed as an operational law on $L^A_{\rm mz}$ as simply:

$$\boxed{L^B_{\rm mz}(m, z) = L^A_{\rm mz}((m - \beta_z\,r)/(\beta_z\,s - \beta_z\,r\,\alpha_z), -\alpha_z).} \tag{10.6}$$

This proves that $\mathbf{B}_N \to B \in \mathcal{M}_{\rm alg}$. □

10.2 $\mathbf{A}_n = \mathbf{X}_{n,N}\mathbf{X}'_{n,N}$, $\mathbf{A}_n \longmapsto \mathbf{B}_N = \mathbf{X}'_{n,N}\mathbf{X}_{n,N}$

Here $\mathbf{X}_{n,N}$ is an $n \times N$ matrix, so that $\mathbf{A}_n$ and $\mathbf{B}_N$ are $n \times n$ and $N \times N$ sized matrices respectively. Let $c_N = n/N$. When $c_N < 1$, $\mathbf{B}_N$ will have $N - n$ eigenvalues of magnitude zero while the remaining n eigenvalues will be identically equal to the eigenvalues of $\mathbf{A}_n$. Thus, the e.d.f. of $\mathbf{B}_N$ is related to the e.d.f. of $\mathbf{A}_n$ as

$$\begin{aligned} F^{\mathbf{B}_N}(x) &= \frac{N - n}{N} I_{[0,\infty)} + \frac{n}{N} F^{\mathbf{A}_n}(x) \\ &= (1 - c_N) I_{[0,\infty)} + c_N\,F^{\mathbf{A}_n}(x). \end{aligned} \tag{10.7}$$

where $I_{[0,\infty)}$ is the indicator function that is equal to 1 when $x \geq 0$.

Similarly, when $c_N > 1$, $\mathbf{A}_n$ will have $n - N$ eigenvalues of magnitude zero while the remaining N eigenvalues will be identically equal to the eigenvalues of $\mathbf{B}_N$. Thus the e.d.f. of $\mathbf{A}_n$ is related to the e.d.f. of $\mathbf{B}_N$ as

$$\begin{aligned} F^{\mathbf{A}_n}(x) &= \frac{n - N}{n} I_{[0,\infty)} + \frac{N}{n} F^{\mathbf{B}_N}(x) \\ &= \left(1 - \frac{1}{c_N}\right) I_{[0,\infty)} + \frac{1}{c_N}\,F^{\mathbf{B}_N}(x). \end{aligned} \tag{10.8}$$

Equation (10.8) is simply (10.7) rearranged; so we do not need to differentiate between the case when $c_N < 1$ and $c_N > 1$.

Thus, as $n, N \to \infty$ with $c_N = n/N \to c$, if $F^{\mathbf{A}_n}$ converges in distribution almost surely (or in probability) to a non-random d.f. F^A, then $F^{\mathbf{B}_N}$ will also convergence in distribution almost surely (or in probability) to a non-random d.f. F^B related to F^A by:

$$F^B(x) = (1-c)I_{[0,\infty)} + c\,F^A(x). \tag{10.9}$$

From (10.9), it is evident that the Stieltjes transform of the limiting spectral measures is related as:

$$m_A(z) = -\left(1 - \frac{1}{c}\right)\frac{1}{z} + \frac{1}{c}m_B(z). \tag{10.10}$$

Rearranging the terms on either side of (10.10) allows us to express $m_B(z)$ in terms of $m_A(z)$ as:

$$m_B(z) = -\frac{1-c}{z} + c\,m_A(z). \tag{10.11}$$

Equation (10.11) can be expressed as an operational law on L^A_{mz} as simply:

$$\boxed{L^B_{\mathrm{mz}}(m,z) = L^A_{\mathrm{mz}}\left(-\left(1-\frac{1}{c}\right)\frac{1}{z} + \frac{1}{c}m, z\right).} \tag{10.12}$$

Thus $\mathbf{B}_N \to B \in \mathcal{M}_{\mathrm{alg}}$. □

10.3 Matrix Subspace Augmentation/Projection

Consider the statement of Theorems 4.1.3 and 4.1.4. By defining $c = n/N$ we can easily derive, as we did above, that $m_A(z)$ can be written in terms of $m_B(z)$ as simply

$$m_A(z) = \left(\frac{1}{c} - 1\right)\frac{1}{\alpha - z} + \frac{1}{c}\,m_A(z). \tag{10.13}$$

This allows us to express $L^B_{\mathrm{mz}}(m,z)$ in terms of $L^A_{\mathrm{mz}}(m,z)$ using the relationship in (10.13) as simply

$$\boxed{L^B_{\mathrm{mz}}(m,z) = L^A_{\mathrm{mz}}\left(-\left(\frac{1}{c}-1\right)\frac{1}{\alpha - z} + \frac{1}{c}\,m, z\right).} \tag{10.14}$$

This proves that $\mathbf{B}_N \to B \in \mathcal{M}_{\mathrm{alg}}$. □

10.4 $\mathbf{A}_N \longmapsto \mathbf{B}_N = \mathbf{A}_N^2$

Here we have $h(x) = x^2$ which is continuous everywhere. From Theorem 10.1, unless $f_A(x)$ has an atomic component at $-s/r$ or s and r are simultaneously zero, $\mathbf{B}_N \to B$. The Stieltjes transform of f_B can be expressed as:

$$m_B(z) = E_Y\left[\frac{1}{y-z}\right] = E_X\left[\frac{1}{x^2-z}\right]. \tag{10.15}$$

Equation (10.15) can be rewritten as:

$$m_B(z) = \frac{1}{2\sqrt{z}}\int \frac{1}{x-\sqrt{z}}f_A(x)dx - \frac{1}{2\sqrt{z}}\int \frac{1}{x+\sqrt{z}}f_A(x)dx = \frac{1}{2\sqrt{z}}m_A(\sqrt{z}) - \frac{1}{2\sqrt{z}}m_A(-\sqrt{z}). \tag{10.16}$$

Equation (10.16) can be expressed as an operational law on the bivariate polynomial L^A_{mz} as simply:

$$\boxed{L^B_{\mathrm{mz}}(m,z) = L^A_{\mathrm{mz}}(2m\sqrt{z},\sqrt{z}) \boxplus_{\mathrm{m}} L^A_{\mathrm{mz}}(-2m\sqrt{z},\sqrt{z}).} \tag{10.17}$$

This proves that $\mathbf{B}_N \to B \in \mathcal{M}_{\mathrm{alg}}$. □

10.5 $\mathbf{A}_N \longmapsto \mathbf{B}_N = \mathbf{A}_N + \mathbf{G}'_{L,N}\,\mathbf{T}_L\,\mathbf{G}_{L,N}$

We prove that $\mathbf{B}_N \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$ by constructing L^B_{mz} using the following theorem by Marčenko-Pastur [10] and Silverstein [15].

Theorem 10.2. *Assume that $\mathbf{G}_{N,L}$ is an $N \times L$ Gaussian-like random matrix. Let $\mathbf{A}_N \xrightarrow{\text{a.s.}} A$ be an $N \times N$ symmetric/Hermitian random matrix and $\mathbf{T}_L \xrightarrow{\text{a.s.}} T$ be an $L \times L$ diagonal atomic random matrix respectively. If $\mathbf{G}_{N,L}$, $\mathbf{A}_N$ and $\mathbf{T}_L$ are independent then $\mathbf{B}_N = \mathbf{A}_N + \mathbf{G}'_{N,L}\mathbf{T}_L\mathbf{G}_{N,L} \xrightarrow{\text{a.s.}} B$, as $c_N = N/L \to c$ for $N, L \to \infty$,. The Stieltjes transform, $m_B(z)$ of the unique spectral measure f_B satisfies*

$$\boxed{m_B(z) = m_A\left(z - c\int \frac{x\, f_T(x)dx}{1 + x\, m_B(z)}\right).} \tag{10.18}$$

Let $\mathbf{T}_L$ be an atomic matrix with d atomic masses of weight p_i and magnitude λ_i for $i = 1, 2, \ldots, d$. From Theorem 10.2, $m_B(z)$ can be written in terms of $m_A(z)$ as simply

$$m_B(z) = m_A\left(z - c\sum_{i=1}^{d} \frac{p_i\,\lambda_i}{1 + \lambda_i\, m_B(z)}\right). \tag{10.19}$$

where we have substituted $f_T(x) = \sum_{i=1}^{d} p_i\,\delta(x - \lambda_i)$ into (10.18) with $\sum_i p_i = 1$.

Equation (10.19) can be expressed as an operational law on the bivariate polynomial L^A_{mz} as simply:

$$\boxed{L^B_{\text{mz}}(m, z) = L^A_{\text{mz}}(m, z - \alpha_m).} \tag{10.20}$$

where $\alpha_m = c\sum_{i=1}^{d} p_i\,\lambda_i/(1 + \lambda_i\, m)$. This proves that $\mathbf{B}_N \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$. □

10.6 $\mathbf{A}_N \longmapsto \mathbf{B}_N = \mathbf{A}_N \times \mathbf{W}_N(c_N)$

We prove that $\mathbf{B}_N \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$ by constructing L^B_{mz} from L^A_{mz} using the following theorem by Bai and Silverstein [2, 15].

Theorem 10.3. *Assume that $\mathbf{W}_N(c_N)$ is an $N \times N$ Wishart-like random matrix. Let $\mathbf{A}_N \xrightarrow{\text{a.s.}} A$ be an $N \times N$ random Hermitian non-negative definite matrix. If $\mathbf{W}_N(c_N)$ and $\mathbf{A}_N$ are independent, then $\mathbf{B}_N = \mathbf{A}_N \times \mathbf{W}_N(c_N) \xrightarrow{\text{a.s.}} B$ as $c_N \to c$. The Stieltjes transform, $m_B(z)$ of the unique spectral measure f_B satisfies*

$$m_B(z) = \int \frac{f_A(x)dx}{\{1 - c - c\,z\,m_B(z)\}x - z}. \tag{10.21}$$

By rearranging the terms in the numeration and denominator, (10.21) can be rewritten as

$$m_B(z) = \frac{1}{1 - c - c\,z\,m_B(z)}\int \frac{f_A(d\tau)}{\tau - \frac{z}{1-c-c\,z\,m_B(z)}}. \tag{10.22}$$

Let $\alpha_{m,z} = 1 - c - c\,z\,m_B(z)$ so that (10.22) can be rewritten as

$$m_B(z) = \frac{1}{\alpha_{m,z}}\int \frac{f_A(x)dx}{x - (z/\alpha_{m,z})}. \tag{10.23}$$

We can express $m_B(z)$ in (10.23) in terms of $m_A(z)$ as simply

$$m_B(z) = \frac{1}{\alpha_{m,z}}\, m_A(z/\alpha_{m,z}). \tag{10.24}$$

Equation (10.24) can, equivalently, also be rewritten as

$$m_A(z/\alpha_{m,z}) = \alpha_{m,z}\, m_B(z). \tag{10.25}$$

Equation (10.25) can be expressed as an operational law on the bivariate polynomial L^A_{mz} as simply:

$$\boxed{L^B_{\text{mz}}(m,z) = L^A_{\text{mz}}(\alpha_{m,z}\, m, z/\alpha_{m,z}).} \tag{10.26}$$

This proves that $\mathbf{B}_N \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$. □

10.7 $\mathbf{A}_N \longmapsto \mathbf{B}_N = (\mathbf{A}_N^{1/2} + \sqrt{s}\,\mathbf{G}_{N,L})(\mathbf{A}_N^{1/2} + \sqrt{s}\,\mathbf{G}_{N,L})'$

We prove that $\mathbf{B}_N \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$ by constructing L^B_{mz} from L^A_{mz} using the following theorem by Dozier and Silverstein [6].

Theorem 10.4. *Assume that $\mathbf{G}_{N,L}$ is an $N \times L$ Gaussian-like random matrix. Let $\mathbf{A}_N \xrightarrow{\text{a.s.}} A$ be an $N \times N$ symmetric/Hermitian random matrix independent of $\mathbf{G}_{N,L}$, $\mathbf{A}_N$. Let $\mathbf{A}_N^{1/2}$ denote an $N \times L$ matrix. If s is a positive real-valued scalar then $\mathbf{B}_N = (\mathbf{A}_N^{1/2} + \sqrt{s}\,\mathbf{G}_{N,L})(\mathbf{A}_N^{1/2} + \sqrt{s}\,\mathbf{G}_{N,L})' \xrightarrow{\text{a.s.}} B$, as $c_N = N/L \to c$ for $N, L \to \infty$. The Stieltjes transform, $m_B(z)$ of the unique spectral measure f_B satisfies*

$$m_B(z) = -\int \frac{f_A(x)dx}{z\,\{1 + s\,c\,m_B(z)\} - \frac{x}{1+s\,c\,m_B(z)} + s\,(c-1)}. \tag{10.27}$$

By rearranging the terms in the numerator and denominator, (10.27) can be rewritten as

$$m_B(z) = \int \frac{\{1 + s\,c\,m_B(z)\}\, f_A(x)dx}{x - \{1 + s\,c\,m_B(z)\}(z\,\{1 + s\,c\,m_B(z)\} + (c-1)\,s)}. \tag{10.28}$$

Let $\alpha_m = 1 + s\,c\,m_B(z)$ and $\beta_m = \{1 + s\,c\,m_B(z)\}(z\,\{1 + s\,c\,m_B(z)\} + (c-1)\,s)$, so that $\beta = \alpha_m^2\, z + \alpha_m\, s(c-1)$. Equation (10.28) can hence be rewritten as

$$m_B(z) = \alpha_m \int \frac{f_A(x)dx}{x - \beta_m}. \tag{10.29}$$

Using the definition of the Stieltjes transform in (6.1), we can express $m_B(z)$ in (10.29) in terms of $m_A(z)$ as simply

$$\begin{aligned} m_B(z) &= \alpha_m\, m_A(\beta_m) \\ &= \alpha_m\, m_A(\alpha_m^2\, z + \alpha_m(c-1)s). \end{aligned} \tag{10.30}$$

Equation (10.30) can, equivalently, be rewritten as

$$m_A(\alpha_m^2\, z + \alpha_m(c-1)s) = \frac{1}{\alpha_m} m_B(z). \tag{10.31}$$

Equation (10.31) can be expressed as an operational law on the bivariate polynomial L_{mz} as simply:

$$\boxed{L^B_{\text{mz}}(m,z) = L^A_{\text{mz}}(m/\alpha_m, \alpha^2\, z + \alpha_m\, s(c-1)).} \tag{10.32}$$

This proves that $\mathbf{B}_N \xrightarrow{\text{a.s.}} B \in \mathcal{M}_{\text{alg}}$. □

10.8 $\mathbf{A}_n, \mathbf{B}_M \longmapsto \mathbf{C_N} = \mathbf{diag}(\mathbf{A}_n, \mathbf{B}_M)$

Let C_N be an $N \times N$ block diagonal matrix formed from the $n \times n$ matrix $\mathbf{A}_n$ and the $M \times M$ matrix $\mathbf{B}_M$. Let $c_N = n/N$. The e.d.f. of $\mathbf{C}_N$ is then simply:

$$F^{\mathbf{C}_N} = c_N \, F^{\mathbf{A}_n} + (1 - c_N) \, F^{\mathbf{B}_M}.$$

Let $n, M \to \infty$ and $n/N \to c$. If $F^{\mathbf{A}_n}$ and $F^{\mathbf{B}_M}$ converge in distribution almost surely (or in probability) to non-random d.f.'s F^A and F^B respectively, then $F^{\mathbf{C}_N}$ will also converge in distribution almost surely (or in probability) to a non-random d.f. F^C given by:

$$F^C = c \, F^A + (1 - c) \, F^B. \tag{10.33}$$

The Stieltjes transform of the spectral measure f_C can hence be written in terms of the Stieltjes transforms of the measures f_A and f_B as:

$$m_C(z) = c \, m_A(z) + (1 - c) \, m_B(z) \tag{10.34}$$

Equation (10.34) can be expressed as an operational law on the bivariate polynomial $L^A_{\text{mz}}(m, z)$ as simply:

$$\boxed{L^C_{\text{mz}} = L^A_{\text{mz}} \left(\frac{m}{c}, z \right) \boxplus_{\text{m}} L^B_{\text{mz}} \left(\frac{m}{1 - c}, z \right).} \tag{10.35}$$

This proves that $\mathbf{C}_N \to C \in \mathcal{M}_{\text{alg}}$. □

10.9 $\mathbf{A}_N, \mathbf{B}_N \longmapsto \mathbf{B}_N = \mathbf{A}_N + \mathbf{Q}_N \mathbf{B}_N \mathbf{Q}'_N$

We prove that $\mathbf{C}_N \xrightarrow{\text{p}} C \in \mathcal{M}_{\text{alg}}$ by constructing L^C_{mz} from L^A_{mz} and L^B_{mz} using the following theorem by Voiculescu [23].

Theorem 10.5. *Let $\mathbf{A}_N \xrightarrow{\text{p}} A$ and $\mathbf{B}_N \xrightarrow{\text{p}} B$ be $N \times N$ hermitian random matrices. Let $\mathbf{Q}_N$ be a Haar distributed unitary/orthogonal matrix independent of $\mathbf{A}_N$ and $\mathbf{B}_N$. Then $\mathbf{C}_N = \mathbf{A}_N + \mathbf{Q}_N \mathbf{B}_N \mathbf{Q}'_N \xrightarrow{\text{p}} C$. The associated spectral measure f_C is the unique probability measure obtained by the free additive convolution of the measures f_A and f_B. Thus $f_C = f_A \boxplus f_B$ where $\boxplus$ denotes free additive convolution which is linearized by the R transform so that*

$$r_C(g) = r_A(g) + r_B(g). \tag{10.36}$$

Equation (10.36) can be expressed as an operational law on the bivariate polynomials L^A_{rg} and L^B_{rg} as simply:

$$\boxed{L^C_{\text{rg}} = L^A_{\text{rg}} \boxplus_{\text{r}} L^B_{\text{rg}}} \tag{10.37}$$

If L_{mz} exists then so does L_{rg} and vice-versa. This proves that $\mathbf{C}_N \xrightarrow{\text{p}} B \in \mathcal{M}_{\text{alg}}$ and shows that the free additive convolution of algebraic probability measures produces an algebraic probability measure. Thus algebraic densities form a semi-group under free additive convolution. □

10.10 $\mathbf{A}_N, \mathbf{B}_N \longmapsto \mathbf{B}_N = \mathbf{A}_N \times \mathbf{Q}_N \mathbf{B}_N \mathbf{Q}'_N$

We prove that $\mathbf{C}_N \xrightarrow{\text{p}} C \in \mathcal{M}_{\text{alg}}$ by constructing L^C_{mz} from L^A_{mz} and L^B_{mz} using the following theorem by Voiculescu [24, 25].

Theorem 10.6. *Let $\mathbf{A}_N \xrightarrow{\text{p}} A$ and $\mathbf{B}_N \xrightarrow{\text{p}} B$ be $N \times N$ hermitian random matrices. Let $\mathbf{Q}_N$ be a Haar distributed unitary/orthogonal matrix independent of $\mathbf{A}_N$ and $\mathbf{B}_N$. Then $\mathbf{C}_N = \mathbf{A}_N \times \mathbf{Q}_N \mathbf{B}_N \mathbf{Q}_N^{'} \xrightarrow{\text{p}} C$ if $\mathbf{C}_N$ has real eigenvalues. The associated spectral measure f_C is the unique probability measure obtained by the free multiplicative convolution of the measures f_A and f_B. Thus $f_C = f_A \boxtimes f_B$ where $\boxtimes$ denotes free multiplicative convolution which can be expressed in terms of the S transform as*

$$s_C(y) = s_A(y) + s_B(y). \tag{10.38}$$

Equation (10.38) can be expressed as an operational law on the bivariate polynomials L_{sy}^A and L_{sy}^B as simply:

$$\boxed{L_{\text{sy}}^C = L_{\text{sy}}^A \boxtimes_{\text{s}} L_{\text{sy}}^B} \tag{10.39}$$

If L_{mz} exists then so does L_{sy} and vice versa. This proves that $\mathbf{B}_N \xrightarrow{\text{p}} B \in \mathcal{M}_{\text{alg}}$ and shows that the free multiplicative convolution of algebraic probability measures, where defined, produces an algebraic probability measure. Thus positive semi-definite algebraic densities form a semi-group under free multiplicative convolution. □

11. Interpreting the root curves of the bivariate polynomial

Consider a bivariate polynomial L_{mz}^A. Let D_m be the degree of $L_{\text{mz}}^A(m,z)$ with respect to m and $l_k(z)$, for $k = 0, \ldots, \text{D}_\text{m}$, be polynomials in z that are the coefficients of m^k. For every z along the real axis, there are at most D_m solutions to the polynomial equation $L_{\text{mz}}^A(m,z) = 0$. The roots of the bivariate polynomial L_{mz}^A define a locus of points (m,z) in $\mathbb{C} \times \mathbb{C}$ referred to as a complex algebraic curve. Since the limiting density is over $\mathbb{R}$, we may simply focus on real values of z.

For each $z \in \mathbb{R}$, there will be D_m values of m "almost always" except at values of z corresponding to *singularities* of $L_{\text{mz}}^A(m,z)$. A singularity occurs at $z = z_0$ if:

- There is a reduction in the degree of m at z_0 so that there are less than D_m roots for $z = z_0$. This occurs when $l_{\text{D}_\text{m}}(z_0) = 0$. Poles of $L_{\text{mz}}^A(m,z)$ occur if some of the m-solutions blow up to infinity.
- There are multiple roots of L_{mz}^A at z_0 so that some of the values of m coalesce.

The singularities constitute the so-called exceptional set of $L_{\text{mz}}^A(m,z)$. Singularity analysis, in the context of algebraic functions, is a well studied problem [8] from which we know that the singularities of $L_{\text{mz}}^A(m,z)$ are constrained to be *branch points*.

A *branch* of the algebraic curve $L_{\text{mz}}^A(m,z) = 0$ is the choice of a locally analytic function $m_j(z)$ defined outside the exceptional set of $L_{\text{mz}}^A(m,z)$ together with a connected region of the $\mathbb{C} \times \mathbb{R}$ plane throughout which this particular choice $m_j(z)$ is analytic. These properties of singularities and branches of algebraic curve are helpful in determining the atomic and non-atomic component of the spectral measure from L_{mz}^A. We note that, as yet, we do not have a fully automated algorithm for extracting the limiting spectral measure from the bivariate polynomial. Development of efficient computational algorithms along the lines described would be of great benefit.

11.1 The atomic component

If there are any atomic components in the limiting spectral measure, they will necessarily manifest themselves as poles of $L_{\text{mz}}^A(m,z)$. This follows from the definition of the Stieltjes transform in (6.1). As mentioned, the poles are located at the roots of $l_{\text{D}_\text{m}}(z)$ which may be computed in Maple as:

```
> Dm := degree(LmzA,m);
> lDmz := coeff(LmzA,m,Dm);
> poles := solve(lDmz=0,z);
```

We can then compute the Puiseux expansion about each of the poles at $z = z_0$. This can be computed in Maple using the `algcurves` package as:

```
> with(algcurves):
> puiseux(Lmz,z=pole,m,1);
```

For the pole at $z = z_0$, we inspect the Puiseux expansions for branches with leading term $1/(z_0 - z)$. An atomic component in the limiting spectrum occurs if and only if the coefficient of such a branch is lesser than or equal one. This constraint ensures that the branch is associated with the Stieltjes transform of a valid probability measure.

Of course, as is often the case with algebraic curves, pathological cases can be easily constructed. For example, more than one branch of the Puiseux expansion might correspond to a candidate atomic component, i.e., they might have weights with the prescribed conditions. In our experimentation, whenever this has happened it has been possible to eliminate the spurious ones by matrix theoretic arguments.

Sometimes it is possible to encounter a double pole at $z = z_0$ corresponding to two valid weights. In such cases, empirical evidence suggests that the branch with the largest coefficient (less than one) is the "right" Puiseux expansion. A argument for picking the "right" atomic weight, based purely on the theory of algebraic curves might be easily answered by experts.

11.2 The non-atomic component

The probability measure can be recovered from the Stieltjes transform by applying the inversion formula in (6.4). Thus, given the bivariate polynomial L^A_{mz}, we can compute all $\mathrm{D_m}$ roots along $z \in \mathbb{R}$ (except at poles or singularities). The non-atomic component of the spectral measure is simply the imaginary part of the "right root" normalized by π. In MATLAB, the $\mathrm{D_m}$ roots can be computed using the sequence of commands:

The limiting spectral measure can be, generically, be analytically expressed nicely when $\mathrm{D_m} = 2$. When using root-finding algorithms, for $\mathrm{D_m} = 2, 3$, the "right root" can often be easily identified; the imaginary branch will always appear with its complex conjugate. The density is just the scaled positive imaginary component. In MATLAB, the command `imag(roots(sym2poly(Lmz)))/pi` will often suffice (except at poles).

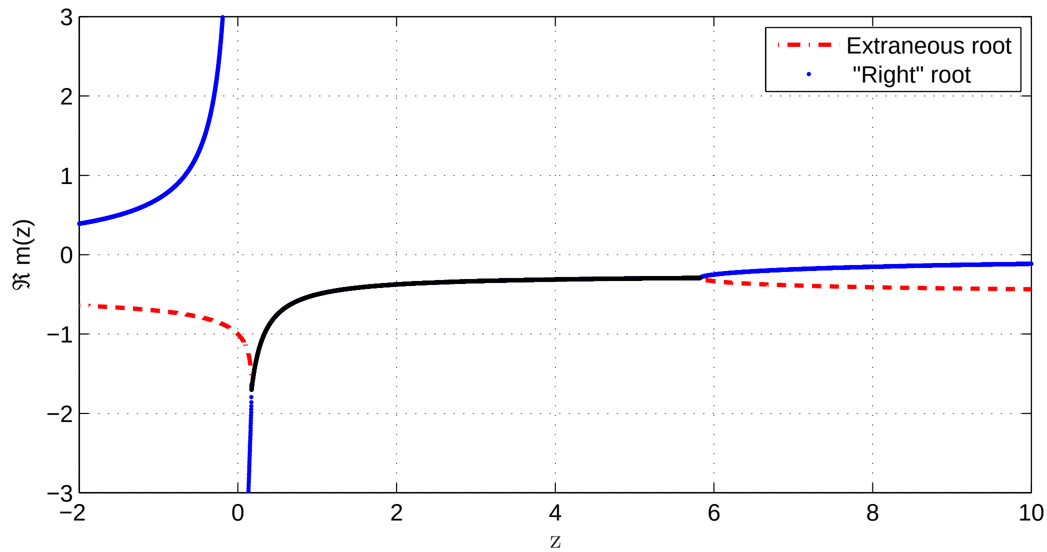
When $\mathrm{D_m} = 4$ onwards, except when L^A_{mz} is bi-quadratic, there is no choice but to manually identify the "right root" among the numerically computed $\mathrm{D_m}$ roots. When the underlying spectral measure is compactly supported, the boundary points will be singularities of the algebraic curve. In particular, when the probability measure is compact and the boundary points are not poles, they occur at points where some values of m coalesce. These points are the roots of the discriminant of L^A_{mz}, computed in Maple as:

```
> PossibleBoundaryPoints = solve(discrim(LmzA,m),z);
```

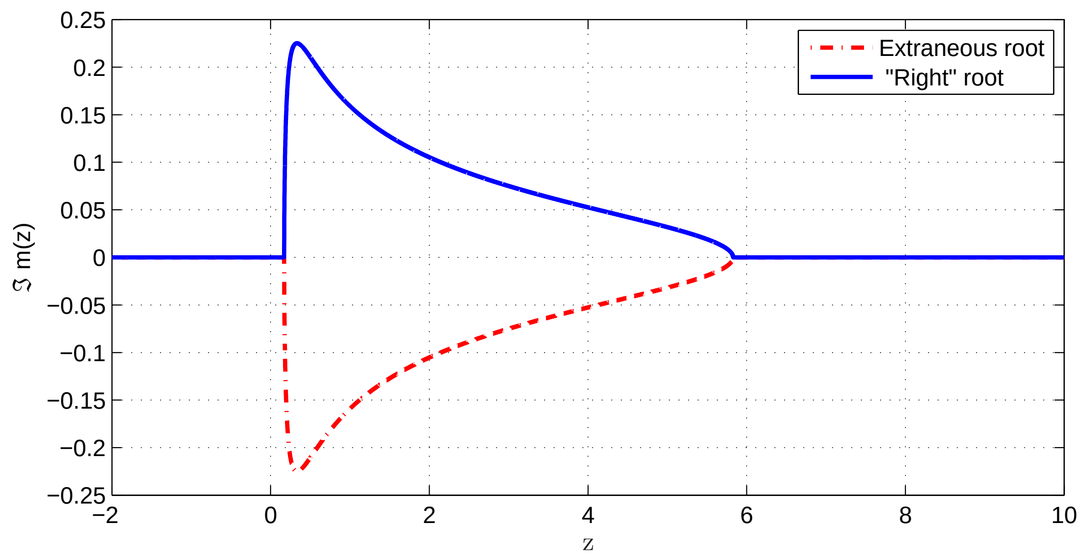
We suspect that "nearly all" compactly supported algebraic random matrices will exhibit a "square root" behavior near boundary points at which there are no poles; this would extend the results in [16]. In the generic case, this will occur whenever the boundary points correspond to locations where two branches of the algebraic curve coalesce.

Irrespective of whether the encoded probability measures is compactly supported or not, the $-1/z$ behavior of the real part of Stieltjes transform (the principal value) as $z \to \pm\infty$ helps identify the correct root. In our experience, while multiple root curves might exhibit this behavior, invariably only one root will have an imaginary branch that, when normalized, will correspond to a valid probability measure. Why this always appears to be the case for the operational laws described is a bit of a mystery to us.

Example: Consider the Marčenko-Pastur density encoded by L_{mz} given in Table 3(b). The Puiseux expansion about the pole at $z = 0$ (the only pole!), has coefficient $(1 - 1/c)$ which corresponds to an atom only when $c > 1$ (as expected using a matrix theoretic argument). Finally, the branch points at $(1 \pm \sqrt{c})^2$ correspond to boundary points of the compactly supported probability measure. Figure 11.2 plots the real and imaginary parts of the algebraic curve for $c = 2$.



(a) Real component. The singularity at zero corresponds to an atom of weight 1/2. The branch points at $(1 \pm \sqrt{2})^2$ correspond to the boundary points of the region of support.



(b) Imaginary component normalized by π. The positive component corresponds to the encoded probability measure.

Figure 3: The real and imaginary components of the algebraic curve defined by the equation $L_{\text{mz}}(m, z) = 0$, where $L_{\text{mz}} \equiv czm^2 - (1 - c - z)\,m + 1$, which encodes the Marčenko-Pastur density. The curve is plotted for $c = 2$. The $-1/z$ behavior of the real part of the "right root" as $z \to \infty$ is the generic behavior exhibited by the real part of the Stieltjes transform of a valid probability measure.

12. Enumerating the moments and free cumulants

In principle, the moments generating function can be extracted from a Puisuex expansion of the algebraic function $L^A_{\mu z}$ about $z = 0$. When the moments of an algebraic probability measure exist, there is additional structure in the moments and cumulants that allows us to enumerate them efficiently. For an algebraic probability measure, we conjecture that the moments of all order exist if and only if the measure is compactly supported.

Definition 12.1 (Algebraic power series). *Let $\mathbb{R}[[x]]$ denote the ring of formal power series in x with real coefficients. A formal power series $v \in \mathbb{R}[[u]]$ is said to be algebraic if there exist polynomials in u, $P_0(u), \ldots, P_{\mathrm{D_u}}(u)$, not all identically zero, such that*

$$P_0(u) + P_1(u)v + \ldots + P_{\mathrm{D_v}}(u)v^{\mathrm{D_v}} = 0.$$

The degree of v is said to be $\mathrm{D_v}$.

Definition 12.2 (D-finite generating functions). *Let $v \in \mathbb{R}[[u]]$. If there exist polynomials $p_0(u), \ldots, p_d(u)$, such that*

$$p_d(u)v^{(d)} + p_{d-1}(u)v^{(d-1)} + \ldots + p_1(u)v^{(1)} + p_0(u) = 0, \tag{12.1}$$

where $v^{(j)} = d^j v/du^j$. Then we say that v is a D-finite (short for differentiably finite) power series. The algebraic function, $v(u)$, is also referred to as a holonomic function.

Definition 12.3 (P-recursive coefficients). *Let a_n for $n \geq 0$ denote the coefficients of a D-finite power series v. If there exist polynomials $P_0, \ldots, P_e \in \mathbb{R}[n]$ with $P_e \neq 0$, such that*

$$P_e(n)a_{n+e} + P_{e-1}(n)a_{n+e-1} + \ldots + p_0(n)a_n = 0,$$

for all $n \in \mathbb{N}$, then the coefficients a_n are said to be P-recursive (short for polynomially recursive).

Theorem 12.4 (Stanley [20]). *Let $v \in \mathbb{R}[[u]]$ be an algebraic power series of degree $\mathrm{D_v}$. Then v is D-finite and satisfies an equation (12.1) of order $\mathrm{D_v}$.*

Proposition 12.5. *If $f_A \in \mathcal{P}_{\mathrm{alg}}$, and the moments exist, then the moment and free cumulant generating functions are algebraic power series. Moreover, both generating functions are D-finite and the coefficients are P-recursive.*

Proof: If $f_A \in \mathcal{P}_{\mathrm{alg}}$, then L^A_{mz} exists. Hence $L^A_{\mu z}$ and L^A_{rg} exist, so that $\mu_A(z)$ and $r_A(g)$ are algebraic power series. By Theorem 12.4 they are D-finite; the moments and free cumulants are hence P-recursive. □

There are powerful symbolic tools available for enumerating the coefficients of algebraic power series. The Maple based package `gfun` is one such example [13]. From the bivariate polynomial $L_{\mu z}$, we can obtain the series expansion up to degree `expansion_degree` by using the command:

```
> with(gfun):
> MomentSeries = algeqtoseries(Lmyuz,z,myu,expansion_degree,'pos_slopes');
```

The option `pos_slopes` computes only those branches tending to zero. Similarly, the free cumulants can be enumerated from L_{rg} using the command:

```
> with(gfun):
> FreeCumulantSeries = algeqtoseries(Lrg,g,r,expansion_degree,'pos_slopes');
```

For computing expansions to a large order, it is best to work with the recurrence relation. For an algebraic power series $v(u)$, the first `number_of_terms` coefficients can be computed from L_{uv} using the sequence of commands:

```
> with(gfun):
> deq := algeqtodiffeq(Luv,v(u));
> rec := diffeqtorec(deq,v(u),a(n));
> p_generator := rectoproc(rec,a(n),list):
> p_generator(number_of_terms);
```

Example: Consider the Marčenko-Pastur density whose bivariate polynomials are listed in Table 3(b). Using the above sequence of commands, we can enumerate the first five terms of the moment generating function:

$$\mu(z) = 1 + z + (c+1)\,z^2 + \left(3\,c + c^2 + 1\right) z^3 + \left(6\,c^2 + c^3 + 6\,c + 1\right) z^4 + O\left(z^5\right).$$

The moment generating function is a D-Finite power series and satisfies a second order differential equation:

$$-z + zc - 1 + (-z - zc + 1)\,\mu\,(z) + \left(z^3c^2 - 2\,z^2c - 2\,z^3c + z - 2\,z^2 + z^3\right)\frac{d}{dz}\mu\,(z) = 0,$$

with initial condition $\mu(0) = 1$. The moments, $M_n = a(n)$, themselves are P-recursive satisfying the recursion:

$$\left(-2\,c + c^2 + 1\right) na\,(n) + ((-2 - 2\,c)\,n - 3\,c - 3)\,a\,(n+1) + (3+n)\,a\,(n+2) = 0$$

with the initial conditions, $a\,(0) = 1, a\,(1) = 1$. The free cumulants can be analogously computed.

What we find rather remarkable is that for many complicated random matrices, it is often possible to enumerate the moments in closed form even when the limiting spectral measure cannot. The linear recurrence satisfied by the moments may be used to analyze their asymptotic growth. When using the sequence of commands described, sometimes more than one solution might emerge. In such cases, we have often found that one can identify the correct moments by checking for the positivity of even moments or the condition $\mu(0) = 1$. More sophisticated techniques might be needed for pathological cases. It might involve verifying that the coefficients enumerated correspond to the moments a positive probability measure.

13. A natural encoding for free convolution

We recall that classical convolution can be expressed entirely as formal power series operations involving the moment generating function. The exponential and ordinary moment generating functions are, with a suitable transformation, simply the Laplace and Stieltjes transforms of the probability measures. Classical additive convolution corresponds to the product of the (formal) exponential moment generating functions. Classical multiplicative convolution corresponds to the coefficient-wise (or Hadamard) product of the (formal) ordinary moment generating functions.

Differentiably finite formal power series are closed under multiplication and Hadamard product. Moreover, if the exponential generating function is D-finite then so is the ordinary generating function. Rational generating functions correspond to probability measures for which the generating function can be written in closed form. They are trivially D-finite. Classical convolution is thus most naturally encoded in the transformations of the linear differential equations satisfied by the Laplace (or Stieltjes) transform. This statement rings particularly true when the transforms cannot be written in closed form.

For free convolution, however, D-finiteness is not the right structure. This is because holonomic functions are closed under functional inversion only when they are algebraic (or rational). The R and S transforms, obtained by a functional inversion of the Stieltjes transform, are key ingredients of free convolution (see Theorems 10.5, 10.6). Hence, even when the ordinary moment generating function is D-finite, the free cumulant generating function (the R transform) will not be – unless it is also algebraic . When the measures being convolved are algebraic, free convolution will also manifest itself as a transformation of the linear differential equations satisfied by the generating functions. Rather, the most natural encoding of free convolution appears to be in the transformations of algebraic equations that the generating functions satisfy.

14. Some applications

14.1 The Jacobi random matrix

The Jacobi matrix ensemble is defined in terms of two independent Wishart matrices $\mathbf{W}_1(c_1)$ and $\mathbf{W}_2(c_2)$ as $\mathbf{J} = (\mathbf{I} + \mathbf{W}_2(c_2)\,\mathbf{W}_1^{-1}(c_1))^{-1}$. The subscripts are not to be confused for the size of the matrices. Listing the computational steps needed to generate a realization of this ensemble, as in Table 11, is the easiest way to identify the sequence of random matrix operations needed to obtain L_{mz}^J. We first start off with

Transformation	Numerical MATLAB code	Symbolic MATLAB code
Initialization	% Pick n, c1, c2 N1=n/c1; N2=n/c2;	% Define symbolic variables syms m c z;
$\mathbf{A}_1 = \mathbf{I}$	A1 = eye(n,n);	Lmz1 = m*(1-z)-1;
$\mathbf{A}_2 = \mathbf{W}_1(c_1) \times \mathbf{A}_1$	G1 = randan(n,N1)/sqrt(N1); W1 = G1*G1'; A2 = W1*A1;	Lmz2 = Antimerism(Lmz1,c1);
$\mathbf{A}_3 = \mathbf{A}_2^{-1}$	A3 = inv(A2);	Lmz3 = invA(Lmz2);
$\mathbf{A}_4 = \mathbf{W}_2(c_2) \times \mathbf{A}_3$	G2 = randan(n,N2)/sqrt(N2); W2 = G2*G2'; A4 = W2*A3;	Lmz4 = Antimerism(Lmz3,c2);
$\mathbf{A}_5 = \mathbf{A}_4 + \mathbf{I}$	A5 = A4+I;	Lmz5 = shiftA(Lmz4,1);
$\mathbf{A}_6 = \mathbf{A}_5^{-1}$	A6 = inv(A5);	Lmz6 = invA(Lmz5);

Table 11: Sequence of MATLAB commands for generating a numerical realization, A_6 of the Jacobi ensemble. The functions used to generate the corresponding bivariate polynomials symbolically are listed in Table 8

.

$\mathbf{A}_1 = \mathbf{I}$. The bivariate polynomial that encodes f_1, applying the definition of the Stieltjes transform in (6.1), is simply:

$$L_{\text{mz}}^1(m,z) \equiv (1-z)m - 1. \tag{14.1}$$

For $\mathbf{A}_2 = \mathbf{W}_1(c_1) \times \mathbf{A}_1$, we can use (10.26) to obtain the bivariate polynomial:

$$L_{\text{mz}}^2(m,z) = z\,c_1\,m^2 - (-c_1 - z + 1)\,m + 1. \tag{14.2}$$

For $\mathbf{A}_3 = \mathbf{A}_2^{-1}$, from (10.6), we obtain the bivariate polynomial:

$$L_{\text{mz}}^3(m,z) = z^2 c_1 m^2 + (c_1\,z + z - 1)\,m + 1. \tag{14.3}$$

For $A_4 = \mathbf{W}_2(c_2) \times \mathbf{A}_3$. We can use (10.26) to obtain the bivariate polynomial:

$$L_{\text{mz}}^4(m,z) = \left(c_1\,z^2 + c_2\,z\right)m^2 + (c_1\,z + z - 1 + c_2)\,m + 1. \tag{14.4}$$

For $\mathbf{A}_5 = \mathbf{A}_4 + \mathbf{I}$, from (10.6), we obtain the bivariate polynomial:

$$L_{\text{mz}}^5(m,z) = \left((z-1)^2\,c_1 + c_2\,\,(z-1)\right)m^2 + (c_1\,\,(z-1) + z - 2 + c_2)\,m + 1. \tag{14.5}$$

Finally, for $\mathbf{J} = \mathbf{A}_6 = \mathbf{A}_5^{-1}$, from (10.6), we obtain the required bivariate polynomial:

$$\begin{aligned} L_{\text{mz}}^J(m,z) \equiv L_{\text{mz}}^6(m,z) &= \left(c_1\,z + z^3 c_1 - 2\,c_1\,z^2 - c_2\,z^3 + c_2\,z^2\right)m^2 \\ &\quad + \left(-1 + 2\,z + c_1 - 3\,c_1\,z + 2\,c_1\,z^2 + c_2\,z - 2\,c_2\,z^2\right)m - c_2\,z - c_1 + 2 + c_1\,z. \end{aligned} \tag{14.6}$$

Using matrix theoretic arguments, it is clear that the random matrix ensembles $\mathbf{A}_3, \ldots \mathbf{A}_6$ are defined only when $c_1 < 1$. There will be an atomic mass of weight $(1 - 1/c_2)$ at 1 whenever $c_2 > 1$. The non-atomic

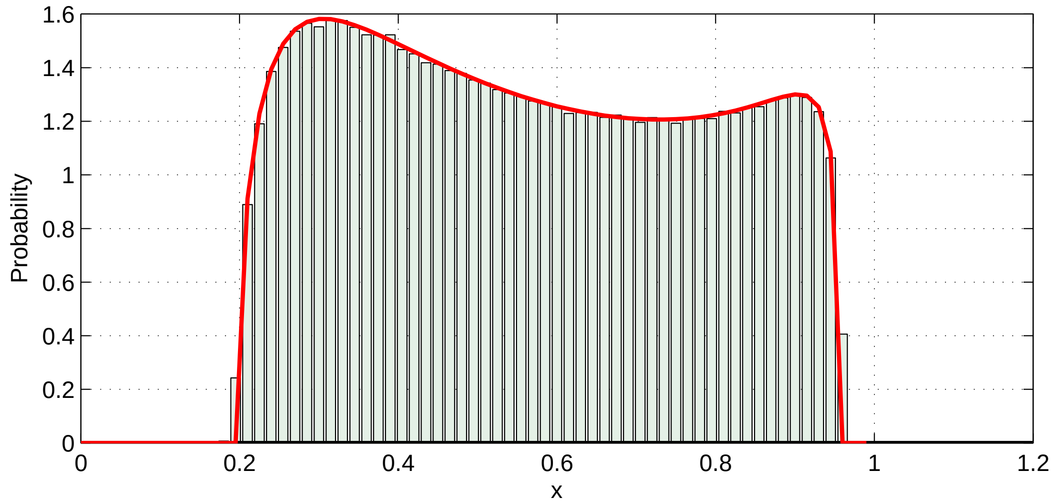


Figure 4: The limiting spectral measure (solid line), $f_{A_6}(x)$, given by (14.7) with $c_1 = 0.1$ and $c_2 = 0.625$ is compared with the normalized histogram of the eigenvalues of a Jacobi matrix generated using the code in Table 11 over 4000 Monte-Carlo trials with $n = 100$, $N_1 = n/c_1 = 1000$ and $N_2 = n/c_2 = 160$.

component of the spectral measure will have a region of support, $\mathcal{S}^- = (a_-, a_+)$. The limiting spectral measure of each of these ensembles can be expressed as:

$$\boxed{f_{A_i}(x) = \frac{\sqrt{(x-a_-)(a_+-x)}}{2\,\pi\, l_2(x)} \qquad \text{for } a_- < x < a_+,} \tag{14.7}$$

for $i = 2, \ldots, 6$, where a_-, a_+ , where the polynomials $l_2(x)$ are listed in Table 12. The moments for

	$l_2(x)$	$a_\pm$
A_2	$x\,c_1$	$(1 \pm \sqrt{c_1})^2$
A_3	$x^2\,c_1$	$\dfrac{1}{(1 \mp \sqrt{c_1})^2}$
A_4	$c_1 x^2 + c_2 x$	$\dfrac{1 + c_1 + c_2 - c_1 c_2 \pm 2\sqrt{c_1 + c_2 - c_1 c_2}}{(1-c_1)^2}$
A_5	$c_1(x-1)^2 + c_2(x-1)$	$\dfrac{c_1^2 - c1 + 2 + c_2 - c_1 c_2 \pm 2\sqrt{c_1 + c_2 - c_1 c_2}}{(1-c_1)^2}$
A_6	$(c_1\,x + x^3 c_1 - 2\,c_1\,x^2 - c_2\,x^3 + c_2\,x^2)$	$\dfrac{(1-c_1)^2}{c_1^2 - c1 + 2 + c_2 - c_1 c_2 \mp 2\sqrt{c_1 + c_2 - c_1 c_2}}$

Table 12: Parameters for determining the spectral measure using (14.7).

the general case when $c_1 \neq c_2$ can be enumerated using the techniques described; they will be quite messy. Instead, consider the special case when $c_1 = c_2 = c$. Using the tools described, the first four terms of the moment series, $\mu(z) = \mu_J(z)$, can be computed directly from $L^J_{\mu z}$ as:

$$\mu(z) = \frac{1}{2} + \left(\frac{1}{8}\,c + \frac{1}{4}\right) z + \left(\frac{3}{16}\,c + \frac{1}{8}\right) z^2 + \left(\frac{1}{32}\,c^2 + \frac{3}{16}\,c - \frac{1}{128}\,c^3 + \frac{1}{16}\right) z^3 + \left(-\frac{5}{256}\,c^3 + \frac{5}{64}\,c^2 + \frac{5}{32}\,c + \frac{1}{32}\right) z^4 + O\left(z^5\right).$$

The moment generating function satisfies the differential equation:

$$-3\,z + 2 + zc + \left(-6\,z^2 + z^3 + 10\,z + z^3 c^2 - 2\,z^3 c - 4\right) \mu\left(z\right)$$
$$+ \left(z^4 - 5\,z^3 - 2\,z^4 c + 8\,z^2 + z^4 c^2 + 2\,z^3 c - 4\,z - z^3 c^2\right) \frac{d}{dz}\mu\left(z\right) = 0, \qquad \text{with } \mu(0) = 1.$$

The moments, $a(n) = M_n$, themselves are P-recursive and obtained by the recursion:

$$\left(-2\,c + c^2 + 1 + \left(-2\,c + c^2 + 1\right) n\right) a\,(n) + \left(\left(-5 + 2\,c - c^2\right) n - 11 + 2\,c - c^2\right) a\,(n+1) + (26 + 8\,n)\,a\,(n+2) + (-16 - 4\,n)\,a\,(n+3) = 0, \qquad \text{with } a(0) = 1/2,\ a(1) = 1/8\,c + 1/4,\ a(2) = 3/16\,c + 1/8.$$

We can similarly compute the recursion for the free cumulants, $a(n) = K_{n+1}$, as:

$$nc^2 a\,(n) + (12 + 4\,n)\,a\,(n+2) = 0, \qquad \text{with } a(0) = 1/2,\ a(1) = 1/8\,c.$$

14.2 Random compression of a matrix

Proposition 14.1. *Let $\mathbf{A}_N \to A \in \mathcal{P}_{\text{alg}}$. Let $\mathbf{Q}_N$ be an $N \times N$ Haar unitary/orthogonal random matrix. Let $\mathbf{B}_n$ be the upper $n \times n$ block of $\mathbf{Q}_N \mathbf{A}_N \mathbf{Q}_N^{'}$. Then $\mathbf{B}_n \to B \in \mathcal{P}_{\text{alg}}$ as $n/N \to c$ for $n, N \to \infty$.*

Proof. Let $\mathbf{P}_N$ be an $N \times N$ projection matrix:

$$\mathbf{P}_N \equiv \mathbf{Q}_N \begin{bmatrix} \mathbf{I}_n & \\ & \mathbf{0}_{N-n} \end{bmatrix} \mathbf{Q}_N^{'}.$$

By definition, $\mathbf{P}_N$ is an atomic matrix so that $\mathbf{P}_N \to P \in \mathcal{M}_{\text{alg}}$ as $n/N \to c$ for $n, N \to \infty$. Let $\widetilde{\mathbf{B}}_N = \mathbf{P}_N \times \mathbf{A}_N$. By Theorem 4.2, $\widetilde{\mathbf{B}}_N \to \widetilde{B} \in \mathcal{M}_{\text{alg}}$. Finally, from Theorem 4.1.2, we have that $\mathbf{B}_n \to B \in \mathcal{M}_{\text{alg}}$. □

The proof above provides a prescription for computing the bivariate polynomial, L^B_{mz}, explicitly as a function of L^A_{mz} and the compression factor c. For this particular application, however, one can use first principles [19] to derive a more direct relationship in terms of the R transform:

$$r_B(g) = r_A(c\,g).$$

This translates into the operational law:

$$\boxed{L^B_{\text{rg}}(r, g) = L^A_{\text{rg}}(r, c\,g).} \tag{14.8}$$

Example: Consider the atomic matrix $\mathbf{A}_N$ half of whose eigenvalues are of magnitude one while the remaining are of magnitude zero. Its limiting spectral measure is given by (7.4). From the bivariate polynomial, L^A_{rg} in Table 3(a) and (14.8), it can be show that the limiting spectral measure of $\mathbf{B}_n$ is encoded by:

$$L^B_{\text{mz}} = \left(-2\,cz^2 + 2\,cz\right) m^2 - (-2\,c + 4\,cz + 1 - 2\,z)\,m - 2\,c + 2,$$

where $n/N \to c$ and $n, N \to \infty$. Poles occur at $z = 0$ and $z = 1$. The leading terms of the Puiseux expansion of the two branches about the poles at $z = z_0$ are:

$$\left\{ \left(\frac{z - z_0}{-2\,c + 4\,c^2} + \frac{1 - 2\,c}{2c} \right) \frac{1}{z - z_0}, \frac{2\,c - 2}{-1 + 2\,c} \right\}.$$

It can be easily seen that when $c > 1/2$, the Puiseux expansion about the poles $z = z_0$ will correspond to an atom of weight $w_0 = (2c - 1)/2c$. Thus the limiting spectral measure is:

$$\boxed{f_B(x) = \max\left(\frac{2c-1}{2c}, 0\right)\delta(x) + \frac{1}{\pi}\frac{\sqrt{(x - a_-)(a_+ - x)}}{2\,xc - 2\,cx^2} I_{[a_-, a_+]} + \max\left(\frac{2c-1}{2c}, 0\right)\delta(x - 1),} \tag{14.9}$$

where $a_\pm = 1/2 \pm \sqrt{-c^2 + c}$. Figure 14.2 compares the theoretical prediction in (14.9) with a Monte-Carlo experiment for $c = 0.4$. From the associated bivariate polynomial:

$$L^B_{\mu z} \equiv (-2\,c + 2\,cz)\,\mu^2 + (z - 2 - 2\,cz + 4\,c)\,\mu - 2\,c + 2,$$

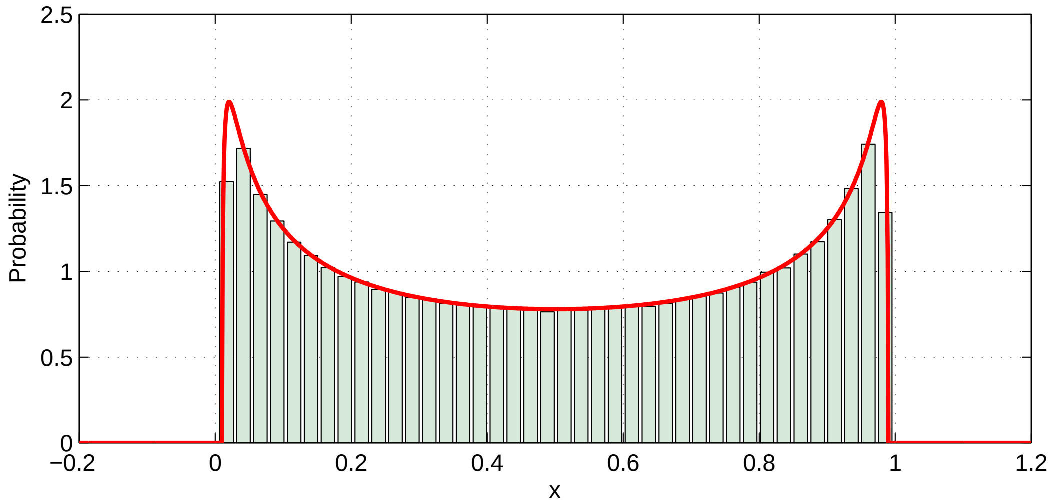


Figure 5: The limiting spectral measure (solid line) of the top $0.4N \times 0.4N$ block of a randomly rotated matrix is compared with the experimental histogram collected over 4000 trials with $N = 200$. Half of the eigenvalues of the original matrix were of magnitude one while the remainder were of magnitude zero.

we obtain two series expansions whose branches tend to zero. The first four terms of the series are given by:

$$1 + \frac{1}{2}z + \frac{1+c}{4}z^2 + \frac{3+c}{8}z^3 + O\left(z^4\right), \tag{14.10}$$

and,

$$\frac{c-1}{c} + \frac{c-1}{2c}z - \frac{(c-1)(-2+c)}{4c}z^2 - \frac{(c-1)(3\,c-4)}{8c}z^3 + O\left(z^4\right), \tag{14.11}$$

respectively. Since $c \leq 1$, the series expansion in (14.11) can be eliminated since $\mu(0) = \int f_B dx \doteq 1$. Thus the coefficients of the series in (14.10) are the moments of the algebraic probability measure f_B. A recursion for the moments can be readily derived using the techniques developed earlier.

14.3 A Wishart random matrix with spatio-temporal correlations

Proposition 14.2. *Assume* $\mathbf{A}_n \xrightarrow{\text{p}} A \in \mathcal{P}_{\text{alg}}^+$, $\mathbf{B}_N \xrightarrow{\text{p}} B \in \mathcal{P}_{\text{alg}}^+$ *and* $\mathbf{G}_{n,N}$ *is an* $n \times N$ *(pure) Gaussian random matrix. Let* $\mathbf{X}_{n,N} = \mathbf{A}_n^{1/2}\mathbf{G}_{n,N}\mathbf{B}_N^{1/2}$. *Then* $\mathbf{C}_n = \mathbf{X}_{n,N}\mathbf{X}_{n,N}^{'} \xrightarrow{\text{p}} C \in \mathcal{P}_{\text{alg}}^+$.

Proof. Let $\mathbf{Y}_{n,N} \equiv \mathbf{G}_{n,N}\mathbf{B}_N^{1/2}$, $\mathbf{T}_n \equiv \mathbf{Y}_{n,N}\mathbf{Y}_{n,N}^{'}$ and $\widetilde{\mathbf{T}}_N = \mathbf{Y}_{n,N}^{'}\mathbf{Y}_{n,N}$. Thus $\mathbf{C}_n = \mathbf{A}_n \times \mathbf{T}_n \equiv \mathbf{A}_n^{1/2} T_n \mathbf{A}_n^{1/2}$. The matrix $\mathbf{T}_n$, as defined, is invariant under orthogonal/unitary transformations, though the matrix $\widetilde{\mathbf{T}}_N$ is not. Hence, by Corollary 4.2, and since $\mathbf{A}_n \to A \in \mathcal{M}_{\text{alg}}$, $\mathbf{C}_N \to C \in \mathcal{M}_{\text{alg}}$ whenever $\mathbf{T}_n \to T \in \mathcal{M}_{\text{alg}}$. From Theorem 4.1.2, $\mathbf{T}_n \to T \in \mathcal{M}_{\text{alg}}$ if $\widetilde{\mathbf{T}}_N \to \widetilde{T} \in \mathcal{M}_{\text{alg}}$. The matrix $\widetilde{\mathbf{T}}_N = \mathbf{B}_N^{1/2}\mathbf{G}_{n,N}^{'}\mathbf{G}_{n,N}\mathbf{B}_N^{1/2}$ is clearly algebraic by application of Corollary 4.2 and Theorem 4.1 since $\mathbf{B}_N$ is algebraic and $\mathbf{G}_{n,N}^{'}\mathbf{G}_{n,N}$ is algebraic and unitarily invariant. Recall that the limiting density of $\mathbf{W}(c_n) = \mathbf{G}_{n,N}\mathbf{G}_{n,N}$ is given by the Marčenko-Pastur density. $\square$

From a computational standpoint, the above proof provides a prescription for computing the bivariate polynomial L_{mz}^{C} using the following sequence of commands:

```
%%%% Assume LmzA and LmzB are given
> syms m z
> LmzW = c*z*m^2-(1-c-z)*m+1; %%%% Encoding of the Marcenko-Pastur density
```

```
> LmzWt = transposeA(b,c);
> LmzTt = AtimesB(LmzWt,LmzB);
> LmzT = transposeA(LmzTt,1/c);
> LmzC = AtimesB(LmzA,LmzT);
```

In statistical applications, n is interpreted as the number of variables (spatial dimension) while N is the number of measurements (temporal dimension). The matrices $\mathbf{A}_n$ and $\mathbf{B}_N$ then model the spatial and temporal covariance structure of the collected data. The "data availability parameter", c, is (roughly speaking) the ratio of the number of variables to the number of measurements. Thus the techniques developed allow us to easily predict (and compute) the limiting spectrum for arbitrary algebraic spatio-temporal covariance structures as a function of the data availability parameter.

Example: Assume the limiting spectral measures of $\mathbf{A}_n$ and $\mathbf{B}_N$ is given by:

$$f_A(x) = f_B(x) = 0.5\,\delta(x-2) + 0.5\,\delta(x-1). \tag{14.12}$$

The Stieltjes transform:

$$m_A(z) = m_B(z) \equiv \frac{0.5}{2-z} + \frac{0.5}{1-z},$$

is a zero of the bivariate polynomial:

$$L_{\mathrm{mz}}^{A} = L_{\mathrm{mz}}^{B} \equiv \left(-6\,z + 2\,z^2 + 4\right) m + 2\,z - 3.$$

Using the sequence of commands above, we can obtain the bivariate polynomial that encodes the limiting spectral measure of $\mathbf{C}_n$. This is given by:

$$L_{\mathrm{mz}}^{C} = \sum_{j=1}^{6} \sum_{k=1}^{4} \left[\mathbf{T}_{\mathrm{mz}}^{C}\right]_{\mathrm{jk}} m^{j-1} z^{k-1},$$

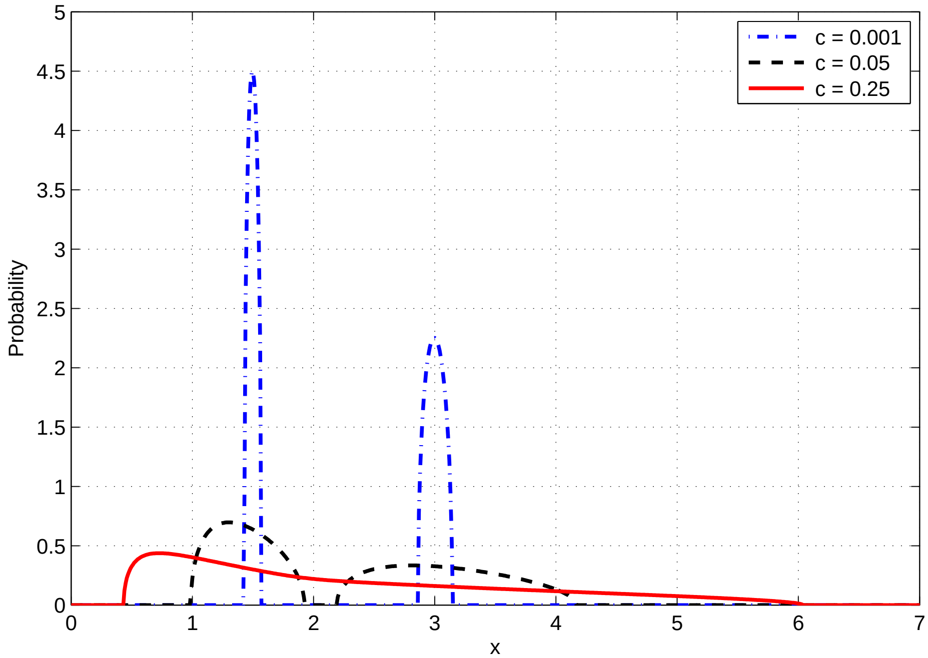
where:

$$\mathbf{T}_{\mathrm{mz}}^{C} \equiv \begin{bmatrix} -18\,c + 18\,c^2 & 18\,c - 9 & 4 & 0 \\ -108\,c^2 + 36\,c + 72\,c^3 & -112\,c + 18 + 130\,c^2 & -18 + 54\,c & 4 \\ 64\,c^2 + 64\,c^4 - 128\,c^3 & 72\,c - 324\,c^2 + 288\,c^3 & 224\,c^2 - 112\,c & 36\,c \\ 0 & 64\,c^2 - 256\,c^3 + 192\,c^4 & 360\,c^3 - 216\,c^2 & 112\,c^2 \\ 0 & 0 & 192\,c^4 - 128\,c^3 & 144\,c^3 \\ 0 & 0 & 0 & 64\,c^4 \end{bmatrix}.$$

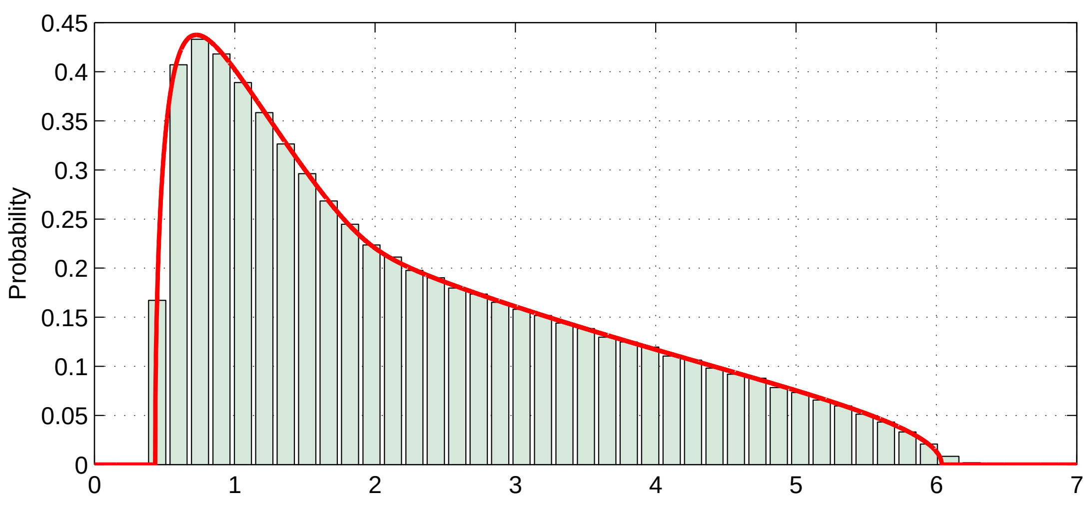
Figure 6(a) plots the limiting spectral measure of $\mathbf{C}_n$ for different values of the data availability parameter. Note how as $c \to 0$, the limiting spectral measure will have two equally weighted atoms at 1.5 and 3. Figure 6(b) compares theory with experiment for $c = 0.25$. Using the sequence of commands described, we can enumerate the first four terms, parameterized by c, of the moment generating function:

$$\mu_C(z) = 1 + \frac{9}{4}z + \left(\frac{45}{8}\,c + \frac{45}{8}\right) z^2 + \left(\frac{675}{16}\,c + \frac{243}{16}\,c^2 + \frac{243}{16}\right) z^3 + \left(\frac{3555}{16}\,c^2 + \frac{1377}{32}\,c^3 + \frac{3555}{16}\,c + \frac{1377}{32}\right) z^4 + O\left(z^5\right).$$

Note how the moments explicitly capture the impact of the data availability parameter, c, on the limiting distribution. This is remarkable, since, for this particular example, it is just not possible to express the spectral measure in closed form.



(a) The limiting spectral measure of $\mathbf{C}_n$ for different values of c. When $c = 0.001$ it means that there are roughly 1000 times as many temporal measurements as there are spatial observations and so on.



(b) The theoretical limiting spectral measure (solid line) for $c = 0.25$ is compared with the normalized histogram of the eigenvalues of C_n collected over 4000 Monte-Carlo trials with $n = 100$ and $N = 400$.

Figure 6: A Wishart random matrix, $\mathbf{C}_n$, with spatio-temporal correlations. The spatial and the temporal covariance matrices have limiting eigenvalue distribution given by (14.12).

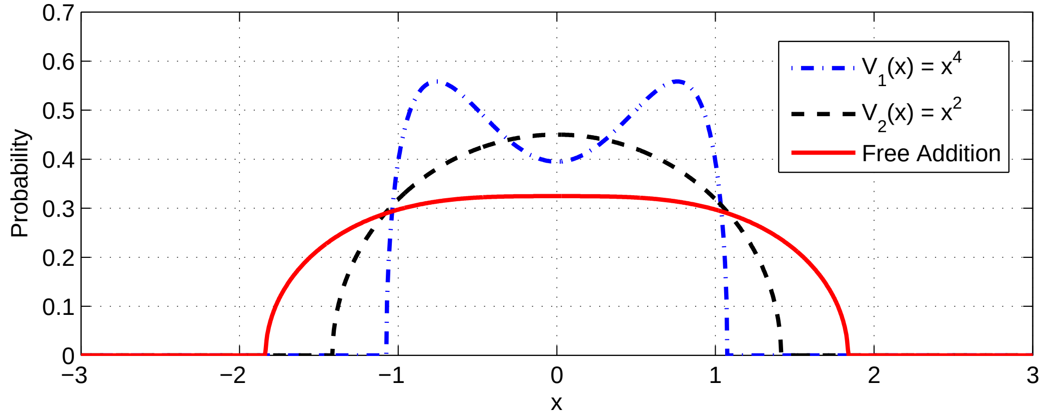


Figure 7: Additive convolution of equilibrium measures corresponding to potentials $V_1(x)$ and $V_2(x)$.

14.4 Free additive convolution of equilibrium measures

Equilibrium measures are a fascinating topic within random matrix theory. They arise in the context of research that examines why very general random models for random matrices exhibit universal behavior in the large matrix limit. Suppose we are given a potential, $V(x)$, then we consider a sequence of Hermitian, unitarily invariant random matrices $\mathbf{A}_N$, the joint distribution of whose elements are of the form:

$$P(\mathbf{A}_N) \propto \exp\left(-N \operatorname{Tr} V(\mathbf{A}_N)\right) d\mathbf{A}_N,$$

where $d\mathbf{A}_N \doteq \prod_{i \leq j} (d\mathbf{A}_N)_{ij}$. The equilibrium measure, when it exists, is the unique probability measure that minimizes the logarithmic energy (see [5] for additional details). The resulting equilibrium measure depends explicitly on the potential $V(x)$ and can be explicitly computed for some potentials. In particular, for potentials of the form $V(x) = t\, x^{2m}$, the Stieltjes transform of the resulting equilibrium measure is an algebraic function [5, Chp. 6.7, pp. 174-175]. In terms of the notation introduced earlier, the equilibrium measure will always be an algebraic probability measure. Hence we can formally investigate the additive convolution of equilibrium measures corresponding to two different potentials. For $V_1(x) = x^2$, the equilibrium measure is the (scaled) semi-circle law encoded by the bivariate polynomial:

$$L^A_{\mathrm{mz}} \equiv m^2 + 2\, m\, z + 2.$$

For $V_2(x) = x^4$, the equilibrium measure is encoded by the bivariate polynomial:

$$L^B_{\mathrm{mz}} \equiv 1/4\, m^2 + m z^3 + z^2 + 2/9\, \sqrt{3}.$$

Since $\mathbf{A}_N$ and $\mathbf{B}_N$ are unitarily invariant random matrices, if $\mathbf{A}_N$ and $\mathbf{B}_N$ are independent, then the limiting spectral measure of $\mathbf{C}_N = \mathbf{A}_N + \mathbf{B}_N$ can be computed from L^A_{mz} and L^B_{mz}. The limiting spectral measure $f_C(x)$ is simply the free additive convolution of f_A and f_B. The MATLAB command `LmzC = AplusB(LmzA,LmzB);` will produce the bivariate polynomial:

$$L^C_{\mathrm{mz}} = -9\, m^4 - 54\, m^3 z + \left(-108\, z^2 - 36\right) m^2 - \left(72\, z^3 + 72\, z\right) m - 72\, z^2 - 16\, \sqrt{3}.$$

Figure 14.4 plots the equilibrium measure for the potentials $V_1(x) = x^2$ and $V_2(x) = x^4$ as well as the free additive convolution of these measures. The interpretation of the resulting measuring in the context of potential theory is not clear. The matrix $\mathbf{C}_N$ will no longer be unitarily invariant so it might not sense to look for a potential $V_3(x)$ for which f_C is an equilibrium measure. The tools and techniques developed in this article might prove useful in further explorations.

14.5 Other applications

There is often a connection between well-known combinatorial numbers and random matrices. For example, the even moments of the Wigner matrix are the famous Catalan numbers. Similarly, if $\mathbf{W}_N(c)$ denotes the Wishart matrix with parameter c, other combinatorial correspondences can be easily established using the techniques developed. For instance, the limiting moments of $\mathbf{W}_N(1) - \mathbf{I}_N$ are the Riordan numbers, the large Schröder numbers correspond to the limiting moments of $2\mathbf{W}_N(0.5)$ while the small Schröder numbers are the limiting moments of $4\mathbf{W}_N(0.125)$. Combinatorial identities along the lines of those developed in [7] might result from these correspondences.

We have successfully extended the techniques developed in this article to characterize a much broader class of random matrices. In forthcoming work we describe how to modify these techniques to compute the limiting distribution of the eigenvalues of commutators of algebraic random matrices, the singular values of the sums of algebraic rectangular random matrices and the eigenvectors of the sums and products of algebraic random matrices.

Documented MATLAB implementation of the polynomial method is available via the RMTool package [12] from `http://www.mit.edu/~raj/rmtool/`; the examples considered in this article, along with many more, appear there.

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