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ON THE SOLUTION OF SOME MINIMAX PROBLEMS *

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Abstract

In dynamic minimax and stochastic optimization problems frequently one is forced to use a suboptimal controller since the computation and implementation of the optimal controller based on dynamic programming is impractical in many cases. In this paper we study the performance of some suboptimal controllers in relation to the performance of the optimal feedback controller and the optimal open-loop controller. Attention is focused on some classes of, so called, open-loop-feedback controllers. It is shown under quite general assumptions that these open-loop-feedback controllers perform at least as well as the optimal open-loop controller. The results are developed for general minimax problems with perfect and imperfect state information. In the latter case the open-loop-feedback controller makes use of an estimator which is required to perform at least as well as a pure predictor in order for the results to hold. Some of the results presented have stochastic counterparts.

1. Introduction

Since the dynamic programming approach towards the optimization of dynamic uncertain systems is often computationally impractical, suboptimal controllers for such systems are in common usage. Such controllers include the optimal open-loop controller, the naive feedback controller, and the open-loop-feedback controller. The precise definition of each of these controllers is not as yet standard in the current literature. For this reason we shall define each of them in relation to a specific minimax problem which will be of continuing interest in this paper.

Problem 1: Given is the uncertain dynamic system

$$x_{k+1} = f_k\ (x_k,u_k,w_k) \qquad k=0,1,.., N-1 \qquad (1)$$

where x_k and u_k denote for all k the state and control of the system and w_k denotes some uncertain parameter. The quantities x_k, u_k and w_k are elements of spaces S_{x_k}, S_{u_k}, S_{w_k}, respectively and the functions $f_k = S_{x_k} \times S_{u_k} \times S_{w_k} \to S_{x_k}$, $k=0,1,.., N-1$ are given. It is assumed that, for each k, the control u_k is constrained to take values from a given subset U_k of S_{u_k}. It is also assumed that the disturbance w_k can take values from a given subset W_k of S_{w_k}. Given the initial state of the system x_o find (if it exists) a control law $\{\mu_o,\mu_1,...,\mu_{N-1}\}$ with $\mu_k = S_{x_k} \to U_k$, $u_k=\mu_k(x_k)$, $k=0,1,..,N-1$ which minimizes the cost functional

$$J(x_o,\mu_o,\mu_1,..,\mu_{N-1}) = \sup_{\substack{w_k \in W_k \\ k=0,1,..,N-1}} g(x_N) \qquad (2)$$

subject to the system equation constraints and where $g=S_{x_N} \to R$ is a given real valued function.

It is worth noting that the above problem formulation is very general since the state, control and disturbance spaces are arbitrary and may differ from one time instant to another. Among other things this allows one to reduce a wide variety of cost functionals to the terminal state cost functional of Eq (2) by means of various state augmentation and transformation techniques.

We shall denote by $J_f\ (x_o)$ the optimal value of the cost functional in Problem 1. This value corresponds to the optimal feedback (o.f.) controller. This controller makes optimal use of the information obtained during the operation of the system, namely the value of the state at each time. The controller which is optimal (assuming it exists) in the class of admissible controllers which ignore the information, i.e., the class of controllers for which $\mu_k(x_k) = c_k$: constant, $\forall\ x_k \in S_{x_k}$ $k=0,1,..,N-1$, is called the optimal open-loop (o.l.) controller and the corresponding value of the cost functional is denoted by $J_{ol}(x_o)$.

Following Witsenhausen [1] we shall call any admissible controller $\{\mu_0,\mu_1,..,\mu_{N-1}\}$ quasi-adaptive if

$$J_f(x_o) \leq J(x_o,\mu_o,\mu_1,..,\mu_{N-1}) \leq J_{ol}(x_o)$$

and adaptive if

$$J_f(x_o) \leq J(x_o,\mu_o,\mu_1,..,\mu_{N-1}) < J_{ol}(x_o)$$

In a practical situation the computation and implementation of the optimal feedback controller is often impractical while the more easily implementable optimal o.l. controller may perform rather poorly. Thus suboptimal adaptive controllers which can be practically implemented are of interest. A suboptimal controller often used in practice is the, so called, naive feedback controller. In order to calculate this controller the disturbance vectors are assumed to have some fixed value $\bar{w}_k$ for each time k with $\bar{w}_k \in W_k$ and the feedback controller which is optimal for the resulting "deterministic" problem is used. This controller need not be calculated by dynamic programming but rather can be implemented by solving an open-loop "deterministic" optimization problem at each time k starting at the observed state x_k. Contrary to deep-rooted convictions among engineers it is known [2] that in general the naive feedback controller may perform strictly worse than the optimal o.l. controller, i.e. it may not be quasi-adaptive.

The open-loop-feedback (o.l.f.) controller [11] which is the main object of study of this paper, is similar to the naive feedback controller except that it takes uncertainty explicitly into account. It is denoted by $\{\tilde{\mu}_o,\tilde{\mu}_1,..,\tilde{\mu}_{N-1}\}$ and defined as follows:

At any time k and state x_k let $\{\tilde{u}_k,\tilde{u}_{k+1},..,\tilde{u}_{N-1}\}$ be the sequence which minimizes (assuming it exists) the cost functional

$$\tilde{J}(x_k,u_k,u_{k+1},..,u_{N-1}) = \sup_{\substack{w \in W_i \\ x_{i+1}=f_i(x_i,u_i,w_i) \\ i=k,..,N-1}} g(x_N)$$

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among all sequences $\{u_k,u_{k+1},..,u_{N-1}\}$ such that $u_i \in U_i, i=k,..,N-1$. The value of the o.l.f. controller at state x_k is given by

$$\tilde{\mu}_k(x_k) = \tilde{u}_k$$

Clearly the o.l.f. controller is easier to implement than the optimal feedback controller and it is more difficult to implement than the optimal o.l. controller (since the optimal o.l. controller is calculated already at the first stage of implementation of the o.l.f. controller). It is a general belief that the o.l.f. controller usually performs considerably better than the optimal o.l. controller. In this paper we prove that the o.l.f. controller performs always at least as well as the optimal o.l. controller, i.e. it is quasi-adaptive. This fact, apparently not proven in the literature even for the stochastic case, is demonstrated in Section 2. In the same section we also consider a problem similar to Problem 1 where in addition there are state constraints. We next consider the case where the controller has imperfect state information. Under these circumstances the o.l.f. controller makes use of an estimator which calculates at each time the set of possible system states given the observations received. This estimator may calculate either the exact set of possible states or a set which bounds the exact set of possible states (presumably this bounding set can be calculated more easily, as is the case, for example, of linear systems with ellipsoidal constraints [3], [4], [5]). It is shown in Section 3 that if the estimator used performs, roughly speaking, better than a pure predictor, then the resulting o.l.f. controller is quasi-adaptive.

2. Performance of the Open-Loop-Feedback Controller for the Perfect State Information Case

Let us assume that the o.l.f. controller $\{\tilde{\mu}_o,\tilde{\mu}_1,..,\tilde{\mu}_{N-1}\}$ exists for Problem 1 and let the corresponding value of the cost functional be denoted by $J_{of}(x_o)$. Then

$$J_{of}(x_o) = J_o(x_o) \tag{3}$$

where the function $J_o: S_{x_o} \to (-\infty,+\infty]$ is given recursively by the algorithm

$$J_{N-1}(x) = \sup_{w \in W_{N-1}} g[f_{N-1}(x,\tilde{\mu}_{N-1}(x),w)] \tag{4}$$

$$J_k(x) = \sup_{w \in W_k} J_{k+1}[f_k(x,\tilde{\mu}_k(x),w)] \qquad k=0,1,..,N-2 \tag{5}$$

Let us also consider the functions $J_k^c: S_{x_k} \to (-\infty,+\infty]$, $k=0,1,..,N-1$, defined by

$$J_k^c(x) = \min_{\substack{u_i \in U_i \\ i=k,k+1,..,N-1}} \sup_{\substack{w_i \in W_i \\ x_{i+1}=f_i(x_i,u_i,w_i) \\ i=k,k+1,..,N-1 \\ x_k=x}} g(x_N) \tag{6}$$

The minimization problem indicated in the above equation must be solved at time k and state x in order to calculate the o.l.f. control. Thus if $\{\tilde{u}_k(x),\tilde{u}_{k+1}(x),..,\tilde{u}_{N-1}(x)\}$ solves the problem in Equ. (6) we have $\tilde{\mu}_k(x)=\tilde{u}_k(x)$. Clearly $J_k^c(x)$ can be interpreted as the calculated open-loop optimal cost from time k to time N and starting from the state x. The optimal open-loop cost is given by

$$J_{ol}(x_o) = J_o^c(x_o)$$

We prove the following proposition:

Proposition 1: For every x and k we have

$$J_k(x) \le J_k^c(x) \tag{7}$$

In particular for $k=0$ we have

$$J_f(x_o) \le J_{of}(x_o) \le J_o^c(x_o) = J_{ol}(x_o) \tag{8}$$

and hence the open-loop-feedback controller is quasi-adaptive.

Proof: We shall prove (7) by induction. Since by the definition of the o.l.f. controller we have $J_{N-1}(x) = J_{N-1}^c(x)$, (7) holds for $k=N-1$. Assume that $J_{k+1}(x) \le J_{k+1}^c(x), \forall x \in S_{x_{k+1}}$. Then for all $x \in S_{x_k}$ and $w \in W_k$ we have

$$J_{k+1}\left[f_k[x,\tilde{\mu}_k(x),w]\right] \le J_{k+1}^c\left[f_k[x,\tilde{\mu}_k(x),w]\right]$$

and

$$J_k(x) = \sup_{w \in W_k} J_{k+1}\left[f_k[x,\tilde{\mu}_k(x),w]\right] \le$$

$$\le \sup_{w \in W_k} J_{k+1}^c\left[f_k[x,\tilde{\mu}_k(x),w]\right] =$$

$$= \sup_{w \in W_k} \min_{\substack{u_i \in U_i \\ i=k+1,k+2,..,N-1}} \sup_{\substack{w_i \in W_i \\ x_{i+1}=f_i(x_i,u_i,w_i) \\ i=k+1,k+2,..,N-1 \\ x_{k+1}=f_k[x,\tilde{\mu}(x),w]}} g(x_N)$$

$$\le \min_{\substack{u_i \in U_i \\ i=k+1,k+2,..,N-1}} \sup_{\substack{w_i \in W_i \\ x_{i+1}=f_i(x_i,u_i,w_i) \\ i=k+1,k+2,..,N-1 \\ x_{k+1}=f_k[x,\tilde{\mu}_k(x),w] \\ w \in W_k}} g(x_N) = J_k^c(x)$$

Q.E.D.

It is to be noted that by (7) the calculated open-loop optimal cost from state x_k, $J_k^c(x_k)$, provides a readily obtainable performance bound for the o.l.f. controller. Two generalizations of the above proposition should also be noted. The first is concerned with the case where the control constraints are state dependent and the disturbance constraints are state and control dependent. In this case the result of Proposition 1 follows by using a similar proof. The second generalization concerns the stochastic version of Problem 1 where the disturbances w_k are random vectors with given probability distribution and the sup in the cost functional (2) is replaced by an expectation. Again a result analogous to the one of Proposition 1 follows by using an entirely similar argument.

The case where in Problem 1 there are additional state constraints $x_k \in X_k$ where X_k is a given subset of S_{x_k} requires separate discussion since one can envisage two different ways of defining the o.l.f. controller. The first method is similar to the one used earlier whereby at state x_k the o.l.f. controller solves the problem

$$\min_{\substack{u_i \in U_i \\ i=k,k+1,..,N-1}} \sup_{\substack{w_i \in W_i \\ x_{i+1}=f_i(x_i,u_i,w_i) \\ i=k,..,N-1}} g(x_N) \tag{9}$$

subject to the additional constraint on the sequence $\{u_k,u_{k+1},..,u_{N-1}\}$ that $x_i \in X_i$, $i=k+1,..,N-1$ for all

$w_i \varepsilon W_i$, $i=k,k+1,..,N-1$. This definition of the o.l.f. controller results in satisfaction of the state constaints (if the o.l.f. controller exists) but does not take into account the fact that the future states will be known to the controller as time progresses. On the other hand an alternate definition of the o.l.f. controller which at first sight appears to offer an advantage takes into account the fact that further information will be received in as much as the satisfaction of the state constraints is concerned. It is known that in order that a feedback controller $\{\mu_o,\mu_1,..,\mu_{N-1}\}$ results in satisfaction of the state constraints $x_i \varepsilon X_i$ it is necessary and sufficient that $\mu_k(x_k) \varepsilon U_k(x_k)$ where $U_k(x_k) \subset U_k$ is a state dependent constraint set implied by the state constraints and generated (possibly at the expense of considerable computation) by means of a recursive algorithm [6],[7],[8]. Hence a possible definition of the o.l.f. controller which results in satisfaction of the state constraints can be given by means of the solution at state x_k of the problem

$$\min_{\substack{U_k \,\varepsilon\, U_k(x_k)\\ U_i \,\varepsilon\, U_i\\ i=k+1,..,N-1}} \; \sup_{\substack{w_i \varepsilon W_i\\ x_{i+1}=f_i(x_i,u_i,w_i)\\ i=k,..,N-1}} g(x_N) \qquad (10)$$

rather than via the solution of problem (9). Clearly the set of admissible control sequences for problem (10) contains the set of admissible control sequences for problem (9) which may be empty even if there exists an admissible control law for the original problem. It would appear that the o.l.f. controller based on the solution of problem (10) performs better than the o.l.f. controller based on problem (9) since for the former the control constraints implied by the state constraints are less strict. This conjecture is not true in general as the following example demonstrates.

Consider the scalar 2-stage system:

$$x_o = 1$$
$$x_1 = x_o+u_o+w_o$$
$$x_2 = f(x_1,u_1)$$

where

$$f(2,0) = f(0,-1) = 0$$
$$f(2,-1) = f(0,0) = 2$$
$$f(1,0) = f(1,-1) = f(-1,0) = f(-1,-1) = 0.5$$

The control constraints are $u_o,u_1 \varepsilon\{0,-1\}$ and the state constraint is $x_2 \varepsilon\{0,0.5\}$. The cost functional is

$$J(x_o,\mu_o,\mu_1) = \max_{w_o\varepsilon\{-1,1\}} x_2$$

It follows by straightforward calculation that the cost corresponding to the o.l.f. controller based on problem (9) is 0 while the cost corresponding to the o.l.f. controller based on problem (10) is 0.5.

The above example demonstrates also a counterintuitive property of the o.l.f. controller for Problem 1 (without state constraints) namely that if the control constraint sets are enlarged it is possible that the performance of the o.l.f. controller deteriorates. This is due, of course, to the fact that the o.l.f. controller does not make optimal use of the additional admissible values.

3. Performance of the Open-Loop-Feedback Controller for the Imperfect Information Case

We turn now to the case where the o.l.f. controller does not have access to the exact system state but rather receives (possible noise-corrupted) measurements providing information about the system state. We consider the following problem.

Problem 2: Consider Problem 1 where the controller has access to measurments of the form

$$z_k = h_k(x_k,v_k) \qquad k=0,1,..,N-1 \qquad (11)$$

where v_k is an uncertain "disturbance" known to belong to a given subset V_k of a space S_{v_k} and the function h_k: $S_{x_k} \times S_{v_k} \to S_{z_k}$ is given. Find (if it exists) a control law $\{\mu_0,\mu_1,..,\mu_{N-1}\}$ with μ_k: $S_{z_o} \times S_{z_1} \times ..\times S_{z_k} \times S_{u_o} \times S_{u_1} \times..\times S_{u_{k-1}} \to U_k$, $u_k = \mu_k(z_o,z_1,..,z_k,u_o,u_1,..,u_{k-1})$, $k=0,1,..,N-1$ which minimizes the cost functional

$$J(x_o,\mu_o,\mu_1,..,\mu_{N-1}) = \sup_{\substack{w_k \,\varepsilon\, W_k\\ v_k \,\varepsilon\, V_k\\ k=0,1,..,N-1}} g(x_N) \qquad (12)$$

Given a particular information sequence $\zeta_k = (z_o,z_1,..,z_k,u_o,u_1,..,u_{k-1})$ there exists a corresponding set of possible states x_k given that ζ_k has occurred. We denote this set by $X_k(\zeta_k)$. It is the set of all states x_k consistent with the measurements $z_o,..,z_k$ the controls $u_o,..,u_{k-1}$ the system equation and the constraints $w_i \varepsilon W_i$, $i=o,..,k-1$, $v_i \varepsilon V_i$, $i=o,1,..,k$. It is known that the set $X_k(\zeta_k)$ constitutes a sufficient information [1],[9],[10] for the controller relative to Problem 2, i.e. the optimal controller need only be a function of $X_k(\zeta_k)$ rather than ζ_k. In a sense that can be well defined the set $X_k(\zeta_k)$ can be viewed as the state of a new system. This system evolves in time according to the equations

$$X_{k+1}(\zeta_{k+1}) = F_k[X_k(\zeta_k),u_k,w_k,v_{k+1}], \; k=0,1,..,N-2 \qquad (13)$$

$$X_N(\zeta_{N-1}) = f_{N-1}[X_{N-1}(\zeta_{N-1}),u_{N-1},w_{N-1}] \qquad (14)$$

$$X_o(\zeta_o) = \{x_o\} \qquad (15)$$

where the function F_k is defined by

$$F_k[X(\zeta_k),u_k,w_k,v_{k+1}] = f_k[X_k(\zeta_k),u_k,w_k] \cap \cap \{x \,|z_{k+1}= h_{k+1}(x,v_{k+1})\} \qquad (16)$$

Let us consider control laws of the form

$\{\sigma_o,\sigma_1,..,\sigma_{N-1}\}$ with σ_k: Subsets of $S_{x_k} \to U_k$, $u_k = \sigma_k[X_k(\zeta_k)]$ and consider also the cost functional

$$J(x_o,\sigma_o,\sigma_1,..,\sigma_{N-1}) = \sup_{\substack{w_k \varepsilon W_k\\ v_k \varepsilon V_k\\ k=0,1,..,N-1}} G[X_N(\zeta_{N-1})] \qquad (17)$$

with

$$G[X_N(\zeta_{N-1})] = \sup_{x_N \,\varepsilon\, X_N(\zeta_{N-1})} g(x_N) \qquad (18)$$

It can be easily seen that Problem 2 is equivalent to the problem of minimizing the cost functional (17) over all admissible control laws $\{\sigma_o,..,\sigma_{N-1}\}$ subject to the system equation constraints (13-16). Since however the controller can calculate (in principle) the set $X_k(\zeta_k)$ this latter problem (call it Problem 2') is one with perfect state information which can be cast within the framework of Problem 1. Now the o.l.f.

controller for Problem 2' is exactly what we shall call the o.l.f. controller for Problem 2. Notice that this o.l.f. controller can be realized by solving at time k the problem

$$\min_{\substack{u_i \in U_i \\ i=k,..,N-1}} \quad \sup_{\substack{x_k \in X_k(\zeta_k) \\ w_i \in W_i \\ x_{i+1}=f_i(x_i,u_i,w_i) \\ i=k,k+1,..,N-1}} g(x_N) \tag{19}$$

and by taking as the current control the first element of the minimizing sequence. Given that the optimal open-loop controller for Problem 2' is the same as the optimal open-loop controller for Problem 2 and using Proposition 1 we have:

<u>Proposition 2</u>: Let $V_{of}(x_o)$, and $V_{ol}(x_o)$ be the values of the cost functional (12) corresponding to the o.l.f. controller and the optimal o.l controller respectively. Then

$$V_{of}(x_o) \le V_{ol}(x_o) \tag{20}$$

While the calculation of the set $X_k(\zeta_k)$ by the controller is possible in principle, in practice this calculation can be very difficult or impossible, i.e. it may be difficult to construct a realization of the corresponding estimator given by Equ.(13) through (16). For this reason it is of interest to examine the performance of open-loop feedback controllers based on estimators that can be more easily implemented. We shall consider a class of estimators which we shall call <u>recursive bounding estimators</u>. Such estimators provide estimate sets

$$\tilde{X}_k(\zeta_k)$$

which contain the set of possible states $X_k(\zeta_k)$ and are realized by a recursive algorithm of the general form

$$\tilde{X}_{k+1}(\zeta_{k+1}) = E_k[\tilde{X}_k(\zeta_k),u_k,z_{k+1}]$$

$$\tilde{X}_o(\zeta_o) = \{x_o\} \tag{21}$$

where E_k is some function. Examples of such recursive bounding estimators are the estimators of [3], [4], [5]. We shall say that a recursive bounding estimator is <u>uncertainty reducing</u> if for each admissible $\zeta_k,u_k,w_k,v_{k+1},k$ we have

$$\tilde{X}_{k+1}(\zeta_{k+1}) = E_k[\tilde{X}_k(\zeta_k),u_k,z_{k+1}] \subset f_k[\tilde{X}_k(\zeta_k),u_k,W_k]$$

Thus the estimator is uncertainty reducing if it provides a set $\tilde{X}_{k+1}(\zeta_{k+1})$ which is contained in the set which would be obtained by pure prediction given $\tilde{X}_k(\zeta_k),u_k$. In other words an uncertainty reducing estimator uses the new measurement z_{k+1} with advantage.

We can define now an o.l.f. controller using a recursive bounding estimator (21) as follows. Given the set $\tilde{X}_k(\zeta_k)$ solve the problem

$$\min_{\substack{u_i \in U_i \\ i \in k,..,N-1}} \quad \sup_{\substack{x_k \in \tilde{X}_k(\zeta_k) \\ w_i \in W_i \\ x_{i+1}= f_i(x_i,u_i,w_i) \\ i=k,..,N-1}} g(x_N) \tag{22}$$

and takes as the current control the first element of the minimizing sequence.

We have the following proposition:

<u>Proposition 3:</u> Let $J_{of}(x_o)$ and $J_{ol}(x_o)$ be the values of the cost functional (12) corresponding to an o.l.f controller using a recursive bounding estimator (21), and to the optimal o.l. controller respectively. If the estimator is uncertainty reducing we have

$$J_{of}(x_o) \le J_{ol}(x_o)$$

Proof: Let $\{\sigma_o,\sigma_1,..,\sigma_{N-1}\}$, $u_k = \sigma_k[\tilde{X}_k(\zeta_k)]$ be the o.l.f. controller. Then $J_{of}(x_o) = J_o(x_o)$ where the function J_o is given recursively by the algorithm

$$J_{N-1}[X_{N-1}(\zeta_{N-1}),\tilde{X}_{N-1}(\zeta_{N-1})] =$$

$$= \sup_{\substack{w \in W_{N-1} \\ x \in X_{N-1}(\zeta_{N-1})}} g\left[f_{N-1}[x,\sigma_{N-1}[\tilde{X}_{N-1}(\zeta_{N-1})],w]\right]$$

$$J_k[X_k(\zeta_k),\tilde{X}_k(\zeta_k)] =$$

$$= \sup_{\substack{w_k \in W_k \\ v_{k+1} \in V_{k+1} \\ z_{k+1} \in h_{k+1}\left[f_k[X_k(\zeta_k),\sigma_k[\tilde{X}_k(\zeta_k)],w_k],v_{k+1}\right]}} J_{k+1}\left[F_k[X_k(\zeta_k),\sigma_k[\tilde{X}_k(\zeta_k)],w_k,v_{k+1}],\; E_k[\tilde{X}_k(\zeta_k),\sigma_k[\tilde{X}_k(\zeta_k)],z_{k+1}]\right]$$

$$J_o(x_o) = \sup_{\substack{w_o \in W_o \\ v_1 \in V_1 \\ z_1 \in h_1[f_k[x_o,\sigma_o(x_o),w_o],v_1]}} J_1\left[F_o[x_o,\sigma_o(x_o),w_k,v_{k+1}],\; E_o[x_o,\sigma_o(x_o),z_1]\right]$$

Consider also (c.f. (6)) the calculated cost

$$J_k^c[X_k(\zeta_k)] = \min_{\substack{u_i \in U_i \\ i=k,..,N-1}} \quad \sup_{\substack{w_i \in W_i \\ x_k \in \tilde{X}(\zeta_k) \\ x_{i+1}=f_i(x_i,u_i,w_i) \\ i=k,..,N-1}} g(x_N)$$

We have $J_o^c(x_o) = J_{ol}(x_o)$.

Thus in order to prove the proposition it will be sufficient to prove that

$$J_k[X_k(\zeta_k),\tilde{X}_k(\zeta_k)] \le J_k^c[\tilde{X}_k(\zeta_k)] \tag{23}$$

for all k and all admissible ζ_k.

By using the fact that the estimator is bounding we have that

$$J_k[X_k(\zeta_k),\tilde{X}_k(\zeta_k)] \le \tilde{J}_k[\tilde{X}_k(\zeta_k)]$$

where $\tilde{J}_k$ is defined for all k by

$$\tilde{J}_{N-1}[\tilde{X}_{N-1}(\zeta_{N-1})] = \sup_{\substack{w \in W_{N-1} \\ x \in \tilde{X}_{N-1}(\zeta_{N-1})}} g\left[f_{N-1}[x,\sigma_{N-1}[\tilde{X}_{N-1}(\zeta_{N-1})],w]\right]$$

$$\tilde{J}_k[\tilde{X}_k(\zeta_k)] = \sup_{z_{k+1} \in h_{k+1}\left[f_k[\tilde{X}_k(\zeta_k),\sigma_k[\tilde{X}_k(\zeta_k)],W_k],V_{k+1}\right]} \tilde{J}_{k+1}\left[E_{k+1}[\tilde{X}_k(\zeta_k),\sigma_k[\tilde{X}_k(\zeta_k)],z_{k+1}]\right]$$

By using the fact that the estimator is uncertainty reducing it follows easily that

$$\tilde{J}_k[\tilde{X}_k(\zeta_k)] \le J_k^c[\tilde{X}_k(\zeta_k)]$$

for all k,ζ_k. Hence (23) holds and the proposition is proved. Q.E.D.

An easily obtained generalization of the above proposition concerns the possibility of proving stochastic counterparts to Propositions 2 and 3. The stochastic counterpart of Prop. 2 can be proved similarly with no difficulty. The role of the set $X_k(\zeta_k)$ is played the the conditional probability $p(x_k \zeta_k)$. However for stochastic problems it is not clear how one is to define the analog of a recursive bounding and uncertainty reducing estimator except for some special cases. One such special case is the well known linear quadratic Gaussian problem. For this case linear estimators will produce Gaussian state estimates which can be partially ordered by means of their error covariance matrix. Thus an uncertainty reducing linear estimator is one for which the corresponding error covariance matrix is smaller (in the pos. definite sense) than the error covariance matrix corresponding to pure prediction. Using this definition a proposition similar to Proposition 3 can be proved.

4. Conclusions

In this paper it was shown under general assumptions that open-loop-feedback controllers perform at least as well as optimal open-loop controllers in dynamic minimax problems. The classes of problems considered include the perfect state information case and the imperfect state information case. In the latter case the open-loop-feedback makes use of an estimator computing either the exact set of possible states of the system or an estimate set that bounds the set of possible states. The estimator is required to perform better than a pure predictor in order for the results to hold. Some of the results in this paper can also be proved in a stochastic control framework.

5. References

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