Polyhedral Approximations in Convex Optimization

Dimitri P. Bertsekas

Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

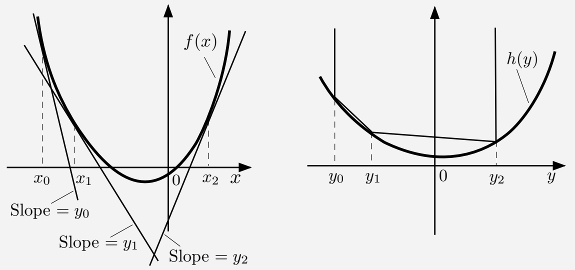
Tsinghua University, 2009

Convex Optimization Algorithms: Generalities

- Primary methodology for large scale problems.
- Arise in the context of duality, network optimization, machine learning.
- Principal methods date to the 60s ... but have been used in new ways recently.
- Descent methods (e.g., subgradient).
- Polyhedral approximation (e.g., cutting plane).
- Proximal (regularization) methods - possibly in combination with polyhedral approximation.

Our Focus in this Talk

- A unifying framework for polyhedral approximation methods.
- Includes classical methods:
  - Cutting plane/Outer linearization
  - Simplicial decomposition/Inner linearization
- Includes new methods, and new versions/extensions of old methods.
- Based on a convex conjugacy framework and outer/inner linearization duality.



Outer Linearization of f Inner Linearization of Conjugate h

Vehicle for Unification

- Extended monotropic programming (EMP)

$$\min_{(x_1,\ldots,x_m)\in S} \sum_{i=1}^{m} f_i(x_i)$$

where $f_i : \Re^{n_i} \mapsto (-\infty, \infty]$ is a convex function and S is a subspace.

- The dual EMP is

$$\min_{(y_1,\ldots,y_m)\in S^\perp} \sum_{i=1}^{m} h_i(y_i)$$

where h_i is the convex conjugate function of f_i.

- Algorithmic Ideas:
  - Outer or inner linearization for some of the f_i
  - Refinement of linearization using duality
- Features of outer or inner linearization use:
  - They are combined in the same algorithm
  - Their roles are reversed in the dual problem
  - Become two (mathematically equivalent dual) faces of the same coin

Advantage over Classical Polyhedral Approximation Methods

- The refinement process is much faster.
  - Reason: At each iteration we add multiple cutting planes/break points (as many as one per function f_i).
  - By contrast a single cutting plane/break point is added in classical methods.
- The refinement process may be more convenient.
  - For example, when f_i is a scalar function, adding a cutting plane/break point to the polyhedral approximation of f_i can be very simple.
  - By contrast, adding a cutting plane/break point may require solving a complicated differentiation/optimization process in classical methods.

References

- D. P. Bertsekas, "Extended Monotropic Programming and Duality," JOTA, 2008, Vol. 139, pp. 209-225.
- D. P. Bertsekas, "Convex Optimization Theory," 2009, www-based "living chapter" on algorithms.
- Related work that applies dual simplicial decomposition in a machine learning context:
  - H. Yu, D. P. Bertsekas, and J. Rousu, "An Efficient Discriminative Training Method for Generative Models," Tech. Report.

$$\min_{x \in \Re^n} \quad r(x) + \sum_{i=1}^{m} f_i(x)$$

  - $f_i(x)$: Complicated polyhedral function; corresponds to ith data batch
  - $r(x)$: Regularization term

Outline

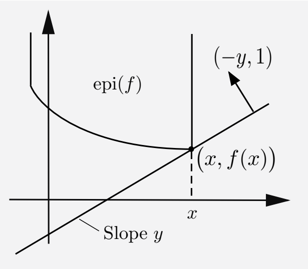
1. Polyhedral Approximation
   - Review of Existing Methodology
   - Cutting Plane and Simplicial Decomposition Methods
2. Extended Monotropic Programming
   - Duality Theory
   - General Approximation Algorithm
3. Special Cases
   - Cutting Plane Methods
   - Simplicial Decomposition for $\min_{x \in C} f(x)$

Subgradients

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be a convex function.
- A vector $y \in \Re^n$ is a *subgradient* of f at a point $x \in \mathrm{dom}(f)$ if

$$f(z) \geq f(x) + y'(z - x), \qquad \forall\, z \in \Re^n$$

- The set $\partial f(x)$ of all subgradients of f at x is the *subdifferential of f at x*
- A subgradient can be identified with a nonvertical supporting hyperplane to the epigraph of f at $\big(x, f(x)\big)$



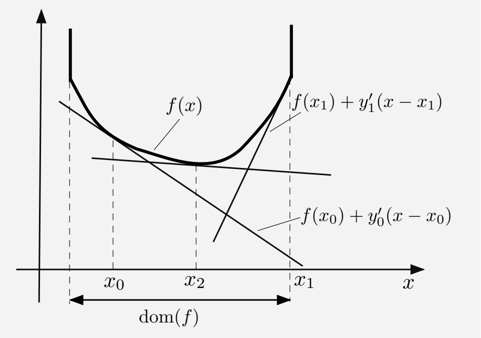
Outer Linearization - Epigraph Approximation by Halfspaces

- Given a convex function $f : \Re^n \mapsto (-\infty, \infty]$.
- Approximation using subgradients:

$$\max\left\{f(x_0) + y_0'(x - x_0), \ldots, f(x_k) + y_k'(x - x_k)\right\}$$

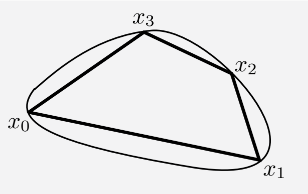
where

$$y_i \in \partial f(x_i), \qquad i = 0, \ldots, k$$

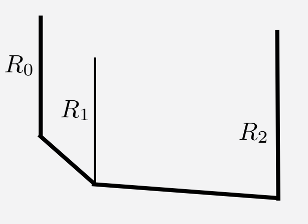


Convex Hulls

- Convex hull of a finite set of points x_i



- Convex hull of a union of a finite number of rays R_i

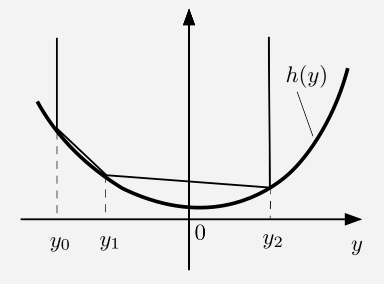


Inner Linearization - Epigraph Approximation by Convex Hulls

- Given a convex function $h : \Re^n \mapsto (-\infty, \infty]$ and a finite set of points

$$y_0, \ldots, y_k \in \mathrm{dom}(h)$$

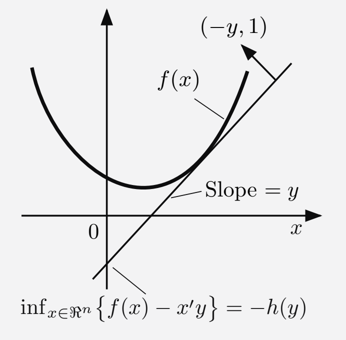
- Epigraph approximation by convex hull of rays $\{(y_i, w) \mid w \geq h(y_i)\}$



Conjugacy

- Consider convex function $f : \Re^n \mapsto (-\infty, \infty]$
- The *conjugate function* of f is

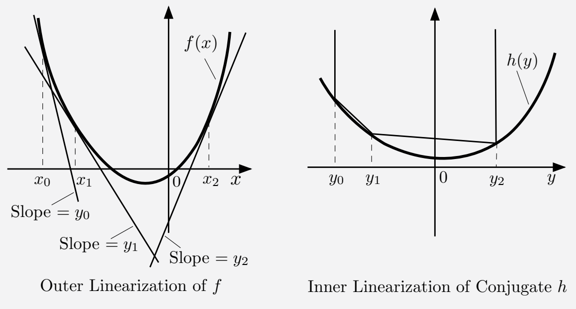
$$h(y) = \sup_{x \in \Re^n} \{x'y - f(x)\}, \qquad y \in \Re^n$$



- Conjugacy theorem: The conjugate of h is f (under very weak conditions)
- Subgradient duality: $y \in \partial f(x)$ iff $x \in \partial h(y)$

Conjugacy of Outer/Inner Linearization

- Given a function $f : \Re^n \mapsto (-\infty, \infty]$ and its conjugate h.
- The conjugate of an outer linearization of f is an inner linearization of h.



- Subgradients in outer lin. <==> Break points in inner lin.

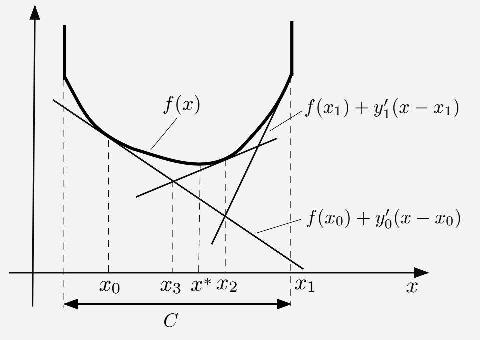
Cutting Plane Method for $\min_{x \in C} f(x)$

- Given $y_i \in \partial f(x_i)$ for $i = 0, \ldots, k$, form

$$F_k(x) = \max \left\{ f(x_0) + y_0'(x - x_0), \ldots, f(x_k) + y_k'(x - x_k) \right\}$$

and let

$$x_{k+1} \in \arg\min_{x \in C} F_k(x)$$



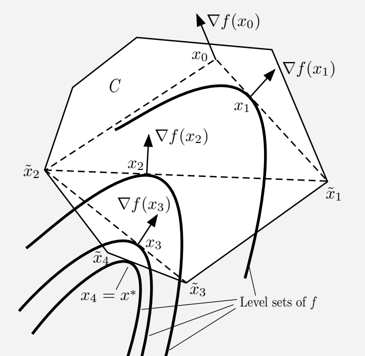
Simplicial Decomposition Method for $\min_{x \in C} f(x)$ (f is Differentiable)

- At the typical iteration we have x_k and $X_k = \{x_0, \tilde{x}_1, \ldots, \tilde{x}_k\}$, where $\tilde{x}_1, \ldots, \tilde{x}_k$ are extreme points of C.
- Generate

$$\tilde{x}_{k+1} \in \arg\min_{x \in X} \{\nabla f(x_k)'(x - x_k)\}$$

- Set $X_{k+1} = \{\tilde{x}_{k+1}\} \cup X_k$, and generate x_{k+1} as

$$x_{k+1} \in \arg \min_{x \in \text{conv}(X_{k+1})} f(x)$$



Comparison: Cutting Plane - Simplicial Decomposition

- Cutting plane aims to use LP with same dimension and smaller number of constraints.
- Most useful when problem has small dimension and:

  There are many linear constraints, or
  The cost function is nonlinear and linear versions of the problem are much simpler

- Simplicial decomposition aims to use NLP over a simplex of small dimension [i.e., $conv(X_k)$].
- Most useful when problem has large dimension and:

  Cost is nonlinear, and
  Solving linear versions of the (large-dimensional) problem is much simpler (possibly due to decomposition)

- The two methods appear very different, with unclear connection, despite the general conjugacy relation between outer and inner linearization.
- We will see that they are special cases of two methods that are dual (and mathematically equivalent) to each other.

Extended Monotropic Programming (EMP)

$$\min_{(x_1,\ldots,x_m)\in S} \sum_{i=1}^{m} f_i(x_i)$$

where $f_i : \Re^{n_i} \mapsto (-\infty, \infty]$ is a closed proper convex, S is subspace.

- Monotropic programming (Rockafellar, Minty), where f_i: scalar functions.
- Single commodity network flow (S: circulation subspace of a graph).
- Block separable problems with linear constraints.
- Fenchel duality framework: Let $m = 2$ and $S = \{(x, x) \mid x \in \Re^n\}$. Then the problem

  $$\min_{(x_1,x_2)\in S} f_1(x_1) + f_2(x_2)$$

  can be written in the Fenchel format

  $$\min_{x\in\Re^n} f_1(x) + f_2(x)$$

- Conic programs (second order, semidefinite - special case of Fenchel).
- Sum of functions (e.g., machine learning): For $S = \{(x, \ldots, x) \mid x \in \Re^n\}$, we obtain

  $$\min_{x\in\Re^n} \sum_{i=1}^{m} f_i(x)$$

Dual EMP

- Derivation: Introduce $z_i \in \Re^{n_i}$ and convert EMP to the equivalent form

$$\min_{\substack{z_i = x_i,\ i=1,\dots,m,\\ (x_1,\dots,x_m)\in S}} \sum_{i=1}^{m} f_i(z_i)$$

- Assign multiplier $y_i \in \Re^{n_i}$ to constraint $z_i = x_i$, and form the Lagrangian

$$L(x,z,y) = \sum_{i=1}^{m} f_i(z_i) + y_i'(x_i - z_i)$$

  where $y = (y_1,\dots,y_m)$.
- The dual problem is to maximize the dual function

$$q(y) = \inf_{(x_1,\dots,x_m)\in S,\ z_i \in \Re^{n_i}} L(x,z,y)$$

- Exploiting the separability of $L(x,z,y)$ and changing sign to convert maximization to minimization, we obtain the dual EMP in symmetric form

$$\min_{(y_1,\dots,y_m)\in S^{\perp}} \sum_{i=1}^{m} h_i(y_i)$$

  where h_i is the convex conjugate function of f_i.

Optimality Conditions

- There are powerful conditions for strong duality $q^* = f^*$ (Bertsekas 2008, generalizing classical monotropic programming results):
  - Vector Sum Condition for Strong Duality: Assume that for all feasible x, the set
  $$S^\perp + \partial_\epsilon (f_1 + \cdots + f_m)(x)$$
  is closed for all $\epsilon > 0$. Then $q^* = f^*$.
  - Special Case: Assume each f_i is finite, or is polyhedral, or is essentially one-dimensional, or is domain one-dimensional. Then $q^* = f^*$.
  - By considering the dual EMP, "finite" may be replaced by "co-finite" in the above statement.

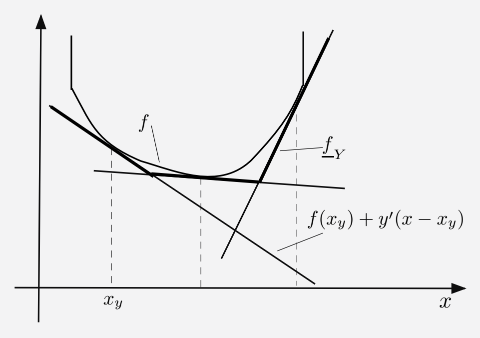
- Optimality conditions, assuming $-\infty < q^* = f^* < \infty$:
  - (x^*, y^*) is an optimal primal and dual solution pair if and only if
  $$x^* \in S, \qquad y^* \in S^\perp, \qquad y_i^* \in \partial f_i(x_i^*), \quad i = 1, \ldots, m$$
  - Symmetric conditions involving the dual EMP:
  $$x^* \in S, \qquad y^* \in S^\perp, \qquad x_i^* \in \partial h_i(y_i^*), \quad i = 1, \ldots, m$$

Outer Linearization of a Convex Function: Definition

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be closed proper convex.
- Given a finite set $Y \subset \mathrm{dom}(h)$, we define the outer linearization of f

$$\underline{f}_Y(x) = \max_{y \in Y} \left\{ f(x_y) + y'(x - x_y) \right\}$$

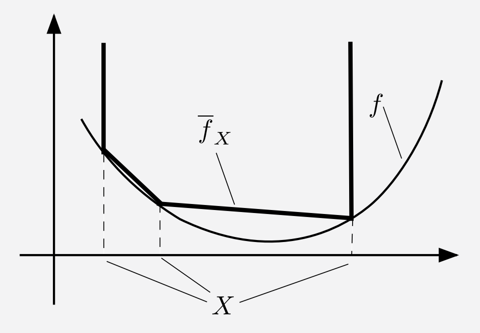
where x_y is such that $y \in \partial f(x_y)$.



Inner Linearization of a Convex Function: Definition

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be closed proper convex.
- Given a finite set $X \subset \mathrm{dom}(f)$, we define the inner linearization of f as the function $\bar{f}_X$ whose epigraph is the convex hull of the rays $\{(x, w) \mid w \geq f(x),\ x \in X\}$:

$$\bar{f}_X(z) = \begin{cases} \min_{\sum_{x \in X} \alpha_x x = z,\ \sum_{x \in X} \alpha_x = 1,\ \alpha_x \geq 0,\ x \in X} \sum_{x \in X} \alpha_z f(z) & \text{if } z \in \mathrm{conv}(X) \\ \infty & \text{otherwise} \end{cases}$$



Polyhedral Approximation Algorithm

- Let $f_i : \Re^{n_i} \mapsto (-\infty, \infty]$ be closed proper convex, with conjugates h_i. Consider the EMP

$$\min_{(x_1,\ldots,x_m)\in S} \sum_{i=1}^{m} f_i(x_i)$$

- Introduce a fixed partition of the index set:

$$\{1,\ldots,m\} = I \cup \underline{I} \cup \bar{I}, \qquad \underline{I} : \text{Outer indices}, \quad \bar{I} : \text{Inner indices}$$

- **Typical Iteration:** We have finite subsets $Y_i \subset \mathrm{dom}(h_i)$ for each $i \in \underline{I}$, and $X_i \subset \mathrm{dom}(f_i)$ for each $i \in \bar{I}$.

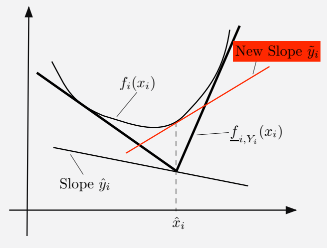
  Find primal-dual optimal pair $\hat{x} = (\hat{x}_1,\ldots,\hat{x}_m)$, and $\hat{y} = (\hat{y}_1,\ldots,\hat{y}_m)$ of the approximate EMP

$$\min_{(x_1,\ldots,x_m)\in S} \sum_{i\in I} f_i(x_i) + \sum_{i\in \underline{I}} \underline{f}_{i,Y_i}(x_i) + \sum_{i\in \bar{I}} \bar{f}_{i,X_i}(x_i)$$

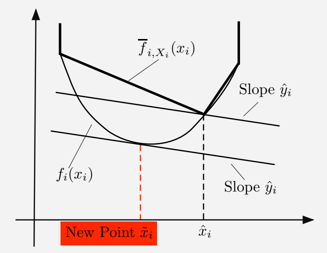
  Enlarge Y_i and X_i by differentiation:
  - For each $i \in \underline{I}$, add $\tilde{y}_i$ to Y_i where $\tilde{y}_i \in \partial f_i(\hat{x}_i)$
  - For each $i \in \bar{I}$, add $\tilde{x}_i$ to X_i where $\tilde{x}_i \in \partial h_i(\hat{y}_i)$.

Enlargement Step for ith Component Function

- Outer: For each $i \in \underline{I}$, add $\tilde{y}_i$ to Y_i where $\tilde{y}_i \in \partial f_i(\hat{x}_i)$.



- Inner: For each $i \in \overline{I}$, add $\tilde{x}_i$ to X_i where $\tilde{x}_i \in \partial h_i(\hat{y}_i)$.



Mathematically Equivalent Dual Algorithm

- Instead of solving the primal approximate EMP

$$\min_{(x_1,\ldots,x_m)\in S} \sum_{i\in I} f_i(x_i) + \sum_{i\in \underline{I}} \underline{f}_{i,Y_i}(x_i) + \sum_{i\in \bar{I}} \bar{f}_{i,X_i}(x_i)$$

  we may solve its dual

$$\min_{(y_1,\ldots,y_m)\in S^\perp} \sum_{i\in I} h_i(y_i) + \sum_{i\in \underline{I}} \underline{h}_{i,Y_i}(y_i) + \sum_{i\in \bar{I}} \bar{h}_{i,X_i}(x_i)$$

  where $\underline{h}_{i,Y_i}$ and $\bar{h}_{i,X_i}$ are the conjugates of $\underline{f}_{i,Y_i}$ and $\bar{f}_{i,X_i}$.
- Note that $\underline{h}_{i,Y_i}$ is an inner linearization, and $\bar{h}_{i,X_i}$ is an outer linearization (roles of inner/outer have been reversed).
- The choice of primal or dual is a matter of computational convenience, but does not affect the primal-dual sequences produced.

Comments on Polyhedral Approximation Algorithm

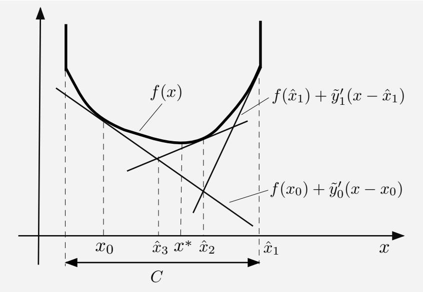
- In some cases we may use an algorithm that solves simultaneously the primal and the dual.
  - **Example:** Monotropic programming, where x_i is one-dimensional.
  - **Special case:** Convex separable network flow, where x_i is the one-dimensional flow of a directed arc of a graph, S is the circulation subspace of the graph.
- In other cases, it may be preferable to focus on solution of either the primal or the dual approximate EMP.
- After solving the primal, the refinement of the approximation ($\tilde{y}_i$ for $i \in \underline{I}$, and $\tilde{x}_i$ for $i \in \bar{I}$) may be found later by differentiation and/or some special procedure/optimization.
  - This may be easy, e.g., in the cutting plane method, or
  - This may be nontrivial, e.g., in the simplicial decomposition method.
- Subgradient duality [$y \in \partial f(x)$ iff $x \in \partial h(y)$] may be useful.

Cutting Plane Method for $\min_{x \in C} f(x)$

- EMP equivalent: $\min_{x_1 = x_2} f(x_1) + \delta(x_2 \mid C)$, where $\delta(x_2 \mid C)$ is the indicator function of C.
- Classical cutting plane algorithm: Outer linearize f only, and solve the primal approximate EMP. It has the form

$$\min_{x \in C} \underline{f}_Y(x)$$

where Y is the set of subgradients of f obtained so far. If $\hat{x}$ is the solution, add to Y a subgradient $\tilde{y} \in \partial f(\hat{x})$.



Simplicial Decomposition Method for $\min_{x \in C} f(x)$

- EMP equivalent: $\min_{x_1 = x_2} f(x_1) + \delta(x_2 \mid C)$, where $\delta(x_2 \mid C)$ is the indicator function of C.
- Generalized Simplicial Decomposition: Inner linearize C only, and solve the primal approximate EMP. In has the form

$$\min_{x \in \bar{C}_X} f(x)$$

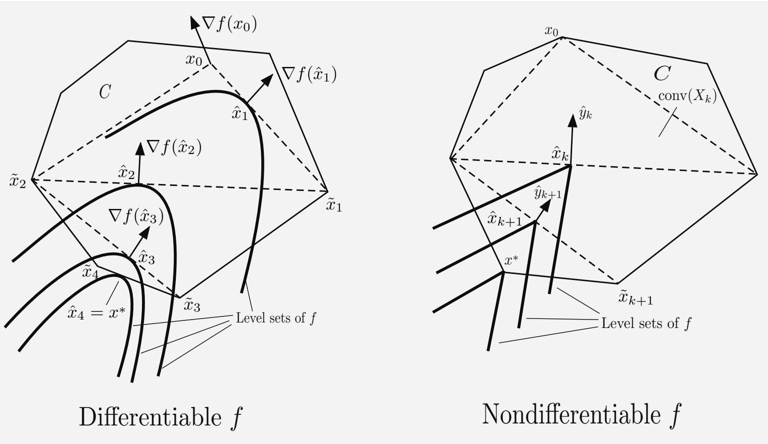
  where $\bar{C}_X$ is an inner approximation to C.
- Assume that $\hat{x}$ is the solution of the approximate EMP.
  - Dual approximate EMP solutions:

$$\{(\hat{y}, -\hat{y}) \mid \hat{y} \in \partial f(\hat{x}),\ -\hat{y} \in (\text{normal cone of } \bar{C}_X \text{ at } \hat{x})\}$$

  - In the classical case where f is differentiable, $\hat{y} = \nabla f(\hat{x})$.
  - Add to X a point $\tilde{x}$ such that

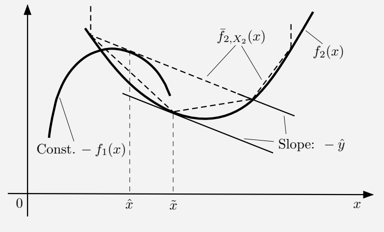
$$\tilde{x} \in \arg\min_{x \in C} \hat{y}' x$$

Illustration of Simplicial Decomposition for $\min_{x \in C} f(x)$

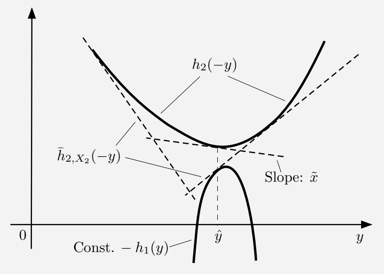


Dual Views for $\min_{x\in\Re^n}\{f_1(x)+f_2(x)\}$

- Inner linearize f_2



- Dual view: Outer linearize h_2



Convergence - Polyhedral Case

- Assume that
  - All outer linearized functions f_i are finite polyhedral
  - All inner linearized functions f_i are co-finite polyhedral
  - The vectors $\tilde{y}_i$ and $\tilde{x}_i$ added to the polyhedral approximations are elements of the finite representations of the corresponding f_i
- Finite convergence: The algorithm terminates with an optimal primal-dual pair.
- Proof sketch: At each iteration two possibilities:
  - Either $(\hat{x}, \hat{y})$ is an optimal primal-dual pair for the original problem
  - Or the approximation of one of the f_i, $i \in \underline{I} \cup \overline{I}$, will be refined/improved
- By assumption there can be only a finite number of refinements. □

Convergence - Pure Cases

- Asymptotic convergence results also available in other special cases, where nonpolyhedral functions f_i are outer/inner linearized.
- Examples are the classical cutting plane and simplicial decomposition methods, and related algorithms.

- Convergence, pure outer linearization ($\bar{I}$: Empty). Assume that the sequence $\{\tilde{y}_i^k\}$ is bounded for every $i \in \underline{I}$. Then every limit point of $\{\hat{x}^k\}$ is primal optimal.
- Proof sketch: For all k, $m \leq k-1$, and $x \in S$, we have

$$\sum_{i \notin \underline{I}} f_i(\hat{x}_i^k) + \sum_{i \in \underline{I}} \left(f_i(\hat{x}_i^m) + (\hat{x}_i^k - \hat{x}_i^m)' \tilde{y}_i^m \right) \leq \sum_{i \notin \underline{I}} f_i(\hat{x}_i^k) + \sum_{i \in \underline{I}} \underline{f}_{i, Y_i^{k-1}}(\hat{x}_i^k) \leq \sum_{i=1}^{m} f_i(x_i)$$

- Let $\{\hat{x}^k\}_{\mathcal{K}} \to \bar{x}$ and take limit as $m \to \infty$, $k \in \mathcal{K}$, $m \in \mathcal{K}$, $m < k$. □

- Exchanging roles of primal and dual, we obtain a convergence result for pure inner linearization case.
- Convergence, pure inner linearization ($\underline{I}$: Empty). Assume that the sequence $\{\tilde{x}_i^k\}$ is bounded for every $i \in \bar{I}$. Then every limit point of $\{\hat{y}^k\}$ is dual optimal.

Concluding Remarks

- A unifying framework for polyhedral approximations based on EMP.
- Dual and symmetric roles for outer and inner approximations.
- There is option to solve the approximation using a primal method or a dual mathematical equivalent - whichever is more convenient/efficient.
- Several classical methods and some new methods are special cases.
- Proximal/bundle-like versions:
  - Convex proximal terms can be easily incorporated for stabilization and for improvement of rate of convergence.
  - Outer/inner approximations can be carried from one proximal iteration to the next.
- Convergence theory so far inspires confidence in the validity of the method.
- More work on complexity/rate of convergence theory is needed.