Incremental Gradient, Subgradient, and Proximal Methods for Convex Optimization

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Problems with Many Additive Cost and Constraint Components

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) \qquad \text{subject to} \qquad x \in X = \cap_{\ell=1}^{q} X_\ell,$$

where $f_i : \Re^n \mapsto \Re$ are convex, and the sets X_ℓ are closed and convex.

Incremental algorithm: Typical iteration

- Choose indexes $i_k \in \{1, \ldots, m\}$ and $\ell_k \in \{1, \ldots, q\}$.
- Perform a subgradient iteration or a proximal iteration

$$x_{k+1} = P_{X_{\ell_k}}\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\big) \quad \text{or} \quad x_{k+1} = \arg\min_{x \in X_{\ell_k}} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \right\}$$

where α_k is a positive stepsize and $\tilde{\nabla}$ denotes (any) subgradient.

Motivation

- Avoid processing all the cost components at each iteration
- Use a simpler constraint to simplify the projection or the proximal minimization

References for this Overview Talk

- Joint and individual works with A. Nedic and M. Wang.
- Focus on convergence, rate of convergence, component formation, and component selection.

- Work on incremental gradient methods and extended Kalman filter for least squares, 1994-1997 (DPB).
- Work on incremental subgradient methods with A. Nedic, 2000-2010.
- Work on incremental proximal methods, 2010-2012 (DPB).
- Work on incremental constraint projection methods with M. Wang 2012-2014 (following work by A. Nedic in 2011).
- See our websites.

Outline

Incremental Subgradient Methods

- Problem: $\min_{x \in X} \sum_{i=1}^{m} f_i(x)$, where f_i and X are convex
- Long history: LMS (Widrow-Hoff, 1960, for linear least squares w/out projection), former Soviet Union literature 1960s, stochastic approximation literature 1960s, neural network literature 1970s

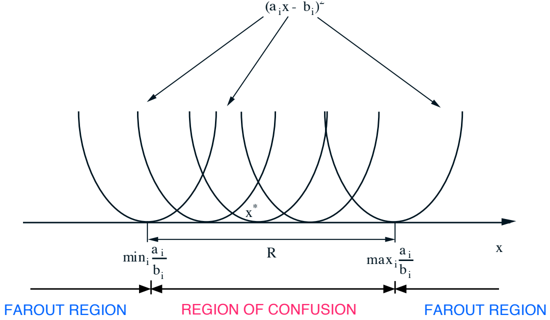
Basic incremental subgradient method

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\big)$$

- Stepsize selection possibilities:
  - $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$
  - α_k: Constant
  - Dynamically chosen (based on estimate of optimal cost)
- Index i_k selection possibilities:
  - Cyclically
  - Fully randomized/equal probability $1/m$
  - Reshuffling/randomization within a cycle (frequent practical choice)

Convergence Mechanism

Quadratic One-Dimensional Example: $\min_{x\in\Re} \sum_{i=1}^{m} (c_i x - b_i)^2$



- Conceptually, the idea generalizes to higher dimensions, but is hard to treat/quantify analytically
- Adapting the stepsize α_k to the farout and confusion regions is an important issue
- Shaping the confusion region is an important issue

Variants of Incremental Subgradient Methods

Method with momentum/extrapolation/heavy ball: $\beta_k \in [0, 1)$

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k) + \beta_k(x_k - x_{k-1})\big)$$

Accelerates in the farout region, decelerates in the confusion region.

Aggregated incremental gradient method

$$x_{k+1} = P_X\left(x_k - \alpha_k \sum_{j=0}^{m-1} \tilde{\nabla} f_{i_{k-j}}(x_{k-j})\right)$$

- Proposed for differentiable f_i, no constraints, cyclic index selection, and constant stepsize, by Blatt, Hero, and Gauchman (2008).
- Recent work by Schmidt, Le Roux, and Bach (2013), randomized index selection, and constant stepsize.
- A fundamentally different convergence mechanism (relies on differentiability and aims at cost function descent). Works even with a constant stepsize (no region of confusion).

Incremental Proximal Methods (DPB, 2010)

Select index i_k and set

$$x_{k+1} = \arg\min_{x \in X} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \right\}$$

Many similarities with incremental subgradient

- Similar stepsize choices
- Similar index selection schemes
- Can be written as

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_{k+1})\big)$$

where $\tilde{\nabla} f_{i_k}(x_{k+1})$ is a special subgradient at x_{k+1} (index advanced by 1)

Compared to incremental subgradient

- Likely more stable
- May be harder to implement

Aggregated Incremental Proximal Method

Select index i_k and set

$$x_{k+1} = \arg\min_{x \in X} \left\{ f_{i_k}(x) + \sum_{j=1}^{m-1} \tilde{\nabla} f_{i_{k-j}}(x_{k-j+1})'(x - x_k) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}$$

where $\tilde{\nabla} f_{i_{k-j}}(x_{k-j+1})$ is special subgradient at x_{k-m+1} (index advanced by 1)

- Can be written as

$$x_{k+1} = P_X \left(x_k - \alpha_k \sum_{j=0}^{m-1} \tilde{\nabla} f_{i_{k-j}}(x_{k-j+1}) \right)$$

- More stable (?) than incremental subgradient or proximal
- May be harder to implement
- Convergence can be shown if $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$

Incremental Subgradient-Proximal Methods

Typical iteration

Choose $i_k \in \{1, \ldots, m\}$ and do a subgradient or a proximal iteration

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\big) \quad \text{or} \quad x_{k+1} = \arg\min_{x \in X} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \right\}$$

where α_k is a positive stepsize and $\tilde{\nabla}$ denotes (any) subgradient.

- Idea: Use proximal when easy to implement; use subgradient otherwise
- A very flexible implementation
- The proximal iterations still require diminishing α_k for convergence

Convergence Analysis

Under Lipschitz continuity-type assumptions:

- Convergence to the optimum for diminishing stepsize.
- Convergence to a neighborhood of the optimum for constant stepsize.
- Faster convergence for randomized index selection (relative to a worst-case cyclic choice).

Notes:

- Fundamentally different from the proximal gradient method, which applies when $m = 2$,

$$\min_{x \in X} \{f_1(x) + f_2(x)\},$$

  and f_1 is differentiable. This is a cost descent method and can use a constant stepsize.
- Aggregated version possible

Incremental Treatment of Many Constraints by Exact Penalties

Problem

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) \qquad \text{subject to} \qquad x \in \cap_{\ell=1}^{q} X_\ell,$$

where $f_i : \Re^n \mapsto \Re$ are convex, and the sets X_ℓ are closed and convex.

Equivalent Problem (Assuming f_i are Lipschitz Continuous)

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) + \gamma \sum_{\ell=1}^{q} \text{dist}(x, X_\ell) \qquad \text{subject to} \qquad x \in \Re^n,$$

where γ is sufficiently large (the two problems have the same set of minima).

Proximal iteration on the dist(x, X_ℓ) function is easy

Project on X_ℓ and interpolate:

$$x_{k+1} = (1 - \beta_k)x_k + \beta_k P_{X_{i_k}}(x_k), \qquad \beta_k = \min\left\{1,\ (\alpha_k\gamma)/\text{dist}(x_k; X_{i_k})\right\}$$

(since γ is large, usually $\beta_k = 1$).

Constraint Projection Methods (Thesis by M. Wang and Joint Papers)

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) \qquad \text{subject to} \qquad x \in \cap_{\ell=1}^{q} X_\ell,$$

where $f_i : \Re^n \mapsto \Re$ are convex, and the sets X_ℓ are closed and convex.

Incremental constraint projection algorithm

- Choose indexes $i_k \in \{1, \ldots, m\}$ and $\ell_k \in \{1, \ldots, q\}$.
- Perform a subgradient iteration or a proximal iteration

$$x_{k+1} = P_{X_{\ell_k}}\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\big) \quad \text{or} \quad x_{k+1} = \arg\min_{x \in X_{\ell_k}} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \right\}$$

where α_k is a positive stepsize and $\tilde{\nabla}$ denotes (any) subgradient.

First proposal and analysis of the case where $m = 1$ and some of the constraints are explicit

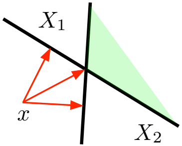
$$X_\ell = \{x \mid g_\ell(x) \leq 0\}$$

was by A. Nedic (2011). Connection to feasibility/alternating projection methods.

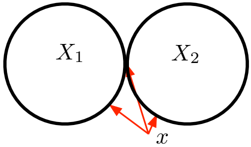
Comparison of the two methods

Second method does not require a penalty parameter γ, but needs a linear regularity assumption: For some $\eta > 0$,

$$\left\| x - P_{\cap_{\ell=1}^{q} X_\ell}(x) \right\| \leq \eta \max_{\ell=1,\ldots,q} \| x - P_{X_\ell}(x) \|, \qquad \forall\, x \in \Re^n$$



Linear Regularity Satisfied



Linear Regularity Violated

Both methods require diminishing stepsize α_k. Unclear how to construct an aggregated version, or any version that is convergent with a constant stepsize.

The second method involves an interesting two-time scale convergence analysis (the subject of the remainder of this talk).

Incremental Random Projection Method

Problem

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) \qquad \text{subject to} \qquad x \in X = \cap_{\ell=1}^{q} X_\ell,$$

Typical iteration

- Choose randomly indexes $i_k \in \{1, \ldots, m\}$ and $\ell_k \in \{1, \ldots, q\}$.
- Set

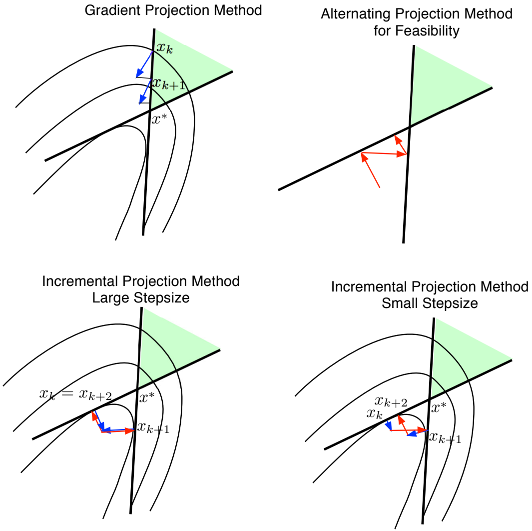
$$x_{k+1} = P_{X_{\ell_k}}\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(\bar{x}_k)\big)$$

- $\bar{x}_k = x_k$ or $\bar{x} = x_{k+1}$.
- $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$ (diminishing stepsize is essential).

Two-way progress

- Progress to feasibility: The projection $P_{X_{\ell_k}}(\cdot)$.
- Progress to optimality: The "subgradient" iteration $x_k - \alpha_k \tilde{\nabla} f_{i_k}(\bar{x}_k)$.

Visualization of Convergence



Progress to feasibility should be faster than progress to optimality. Gradient stepsizes α_k should be $<<$ than the feasibility stepsize of 1.

Sampling Schemes for Constraint Index ℓ_k

Nearly independent sampling

$$\inf_{k \geq 0} \text{Prob}(\ell_k = X_\ell \mid \mathcal{F}_k) > 0, \qquad \ell = 1, \ldots, q,$$

where $\mathcal{F}_k$ is the history of the algorithm up to time k.

Cyclic sampling

Deterministic or random reshuffling every q iterations.

Most distant constraint sampling

$$\ell_k = \arg \max_{\ell = 1, \ldots, q} \left\| x_k - P_{X_\ell}(x_k) \right\|$$

Markov sampling

Generate ℓ_k as the state of an ergodic Markov chain with states $1, \ldots, q$.

Sampling Schemes for Cost Component Index i_k

Random independent uniform sampling

Each index $i \in \{1, \ldots, m\}$ is chosen with equal probability $1/m$, independently of earlier choices.

Cyclic sampling

Deterministic or random reshuffling every m iterations.

Markov sampling

Generate i_k as the state of a Markov chain with states $1, \ldots, m$, and steady state distribution $\{1/m, \ldots, 1/m\}$.

Convergence Theorem

Assuming Lipschitz continuity of the cost, linear regularity of the constraint, and nonemptiness of the optimal solution set, $\{x_k\}$ converges to some optimal solution x^* w.p. 1, under any combination of the preceding sampling schemes.

Idea of the convergence proof

There are two convergence processes taking place:

- Progress towards feasibility, which is fast (geometric thanks to the linear regularity assumption).
- Progress towards optimality, which is slower (because of the diminishing stepsize α_k).
- This two-time scale convergence analysis idea is encoded in a coupled supermartingale convergence theorem, which governs the evolution of two measures of progress

$$\mathbf{E}[\text{dist}^2(x_k, X)] : \text{ Distance to the constraint set, which is fast}$$

$$\mathbf{E}[\text{dist}^2(x_k, X^*)] : \text{ Distance to the optimal solution set, which is slow}$$

Concluding Remarks

- Incremental methods exhibit interesting and complicated convergence behavior
- Proximal variants enhance reliability
- Constraint projection variants provide flexibility and enlarge the range of potential applications
- Issues not discussed:
  - Distributed asynchronous implementation
  - Incremental Gauss-Newton methods (Extended Kalman Filter)

Thank you!