Some New Directions in Dynamic Programming with Cost Function Approximation

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Outline

1. Three Interrelated Research Directions
   - Seminorm Projections (Unifying Projected Equation and Aggregation Approaches)
   - Generalized Bellman Equations (Multistep with State-Dependent Weights)
   - Free Form Sampling (A Flexible Alternative to Single Long Trajectory Simulation)
2. Aggregation and Seminorm Projected Equations
3. Simulation-Based Solution
   - Iterative and Matrix Inversion Methods
   - Free-Form Sampling

Bellman Equations and their Fixed Points

Bellman equation for a policy μ of an n-state α-discounted MDP

$$J = T_\mu J$$

where

$$(T_\mu J)(i) \stackrel{\text{def}}{=} \sum_{j=1}^{n} p_{ij}\big(\mu(i)\big)\big(g(i,\mu(i),j) + \alpha J(j)\big), \qquad i = 1, \ldots, n$$

$p_{ij}(\mu(i))$: transition probs, $g(i,\mu(i),j)$: cost per stage for μ

Bellman equation for the optimal cost function of an n-state MDP

$$J = TJ$$

where

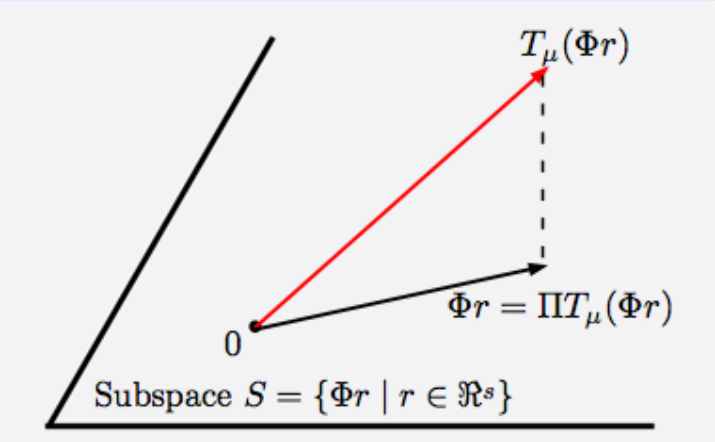
$$(TJ)(i) \stackrel{\text{def}}{=} \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J(j)\big), \qquad i = 1, \ldots, n$$

$p_{ij}(u)$: transition probs, $g(i,u,j)$: cost per stage for a control u

Subspace Approximation $J \approx \Phi r$ (Using a Matrix of Basis Functions Φ)

Methods with subspace approximation

- Projected equation (Galerkin) approach $\Phi r = \Pi T_\mu(\Phi r)$ (Π is projection with respect to some weighted Euclidean norm)
- Aggregation approach $\Phi r = \Phi D T_\mu(\Phi r)$ (Φ and D are matrices whose rows are probability distributions)
- Bellman error method ($\Phi r = \Pi \hat{T}_\mu(\Phi r)$, for a modified mapping $\hat{T}_\mu$ that has the same fixed points as T_μ)



First direction of research aims to connect all these

All of these can be written as $\Phi r = \Pi T_\mu(\Phi r)$, where Π is a seminorm weighted Euclidean projection

Another Direction of Research: Generalized Bellman Equations

Ordinary Bellman equation for a policy μ of an n-state MDP

$$J = T_\mu J$$

Generalized Bellman equation

$$J = T_\mu^{(w)} J$$

where w is a matrix of weights $w_{i\ell}$:

$$(T_\mu^{(w)} J)(i) \stackrel{\text{def}}{=} \sum_{\ell=1}^{\infty} w_{i\ell} (T_\mu^\ell J)(i), \qquad w_{i\ell} \geq 0, \ \sum_{\ell=1}^{\infty} w_{i\ell} = 1 \quad \text{(for each } i = 1, \ldots, n)$$

Both can be solved for J_μ, the cost vector of policy μ.

Two differences of generalized vs ordinary Bellman equations

- Multistep mappings (an old idea, e.g., TD(λ))
- State dependent weights (a new idea)

Special Cases

Classical TD(λ) mapping, $\lambda \in [0, 1)$

$$T^{(\lambda)}J = (1-\lambda)\sum_{\ell=1}^{\infty}\lambda^{\ell-1}T^{\ell}J, \qquad w_{i\ell} = (1-\lambda)\lambda^{\ell-1}$$

A generalization: State-dependent $\lambda_i \in [0, 1)$

$$(T^{(w)}J)(i) = (1-\lambda_i)\sum_{\ell=1}^{\infty}\lambda_i^{\ell-1}(T^{\ell}J)(i), \qquad w_{i\ell} = (1-\lambda_i)\lambda_i^{\ell-1}$$

Why state dependent weights?

- They may allow exploitation of prior knowledge for better approximation (emphasize important states)
- They may facilitate simulation (for special cases such as aggregation)

A Third Direction for Research: Flexible/Free-Form Simulation

Classical TD Sampling

$$T^{(\lambda)}J = (1-\lambda)\sum_{\ell=1}^{\infty}\lambda^{\ell-1}T^{\ell}J$$

- Simulate one single infinitely long trajectory, and move the starting state to generate multiple (infinitely long) trajectories
- This is well-matched to the structure of TD
- Does not work well in the aggregation context, where there are both regular and aggregate transitions (powers $T^{\ell}J$ involve ℓ regular transitions but no aggregate transitions)
- TD sampling matches well with regular transitions but not with aggregate transitions

Free-form sampling

- Generates many short trajectories (length ℓ $<->$ term $T^{\ell}J$)
- Arbitrary restart distribution
- Connects well with state-dependent weights (and allows restarting at an aggregate state in the case of aggregation)

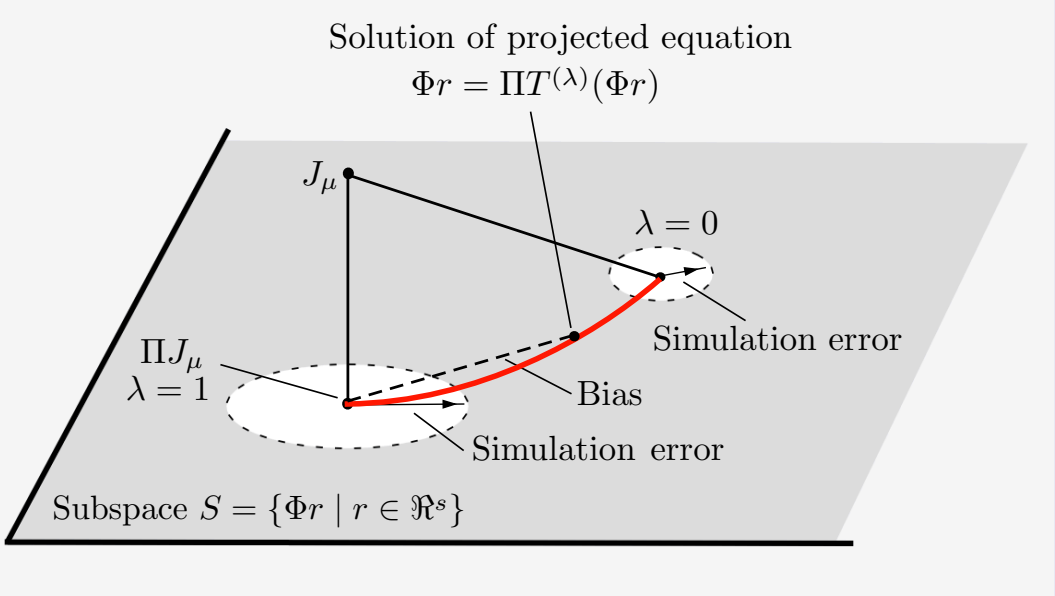
References

- D. P. Bertsekas, "λ-Policy Iteration: A Review and a New Implementation," in *Reinforcement Learning and Approximate Dynamic Programming for Feedback Control*, by F. Lewis and D. Liu (eds.), IEEE Press, Computational Intelligence Series, 2012 (simulation with short trajectories and restart, as a means to control exploration).
- H. Yu and D. P. Bertsekas, "Weighted Bellman Equations and their Applications in Approximate Dynamic Programming," Report LIDS-P-2876, MIT, 2012 (weighted Bellman equations and seminorm projections).
- D. P. Bertsekas, Dynamic Programming and Optimal Control, Vol. II, 4th Edition: Approximate Dynamic Programming, Athena Scientific, Belmont, MA, 2012 (a general reference where all the ideas are mentioned with limited analysis).

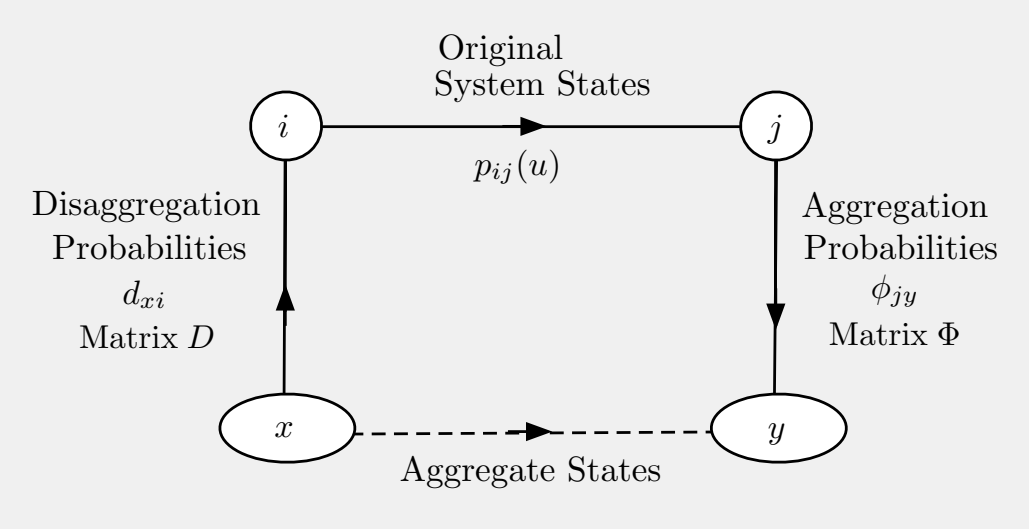
Generalized Bellman Eqs with Seminorm Projection: $\Phi r = \Pi T^{(w)}(\Phi r)$

- Φ is an $n \times s$ matrix of features, defining subspace $S = \{\Phi r \mid r \in \Re^s\}$, $r \in \Re^s$ is a vector of weights.
- Π is projection onto S with respect to a weighted Euclidean seminorm $\|J\|_\xi^2 = \sum_{i=1}^n \xi_i \big(J(i)\big)^2$, where $\xi = (\xi_1, \ldots, \xi_n)$, with $\xi_i \geq 0$.
- Bias-variance tradeoff applies to both norm and seminorm cases.

Example: TD(λ) $T^{(\lambda)} J = (1-\lambda) \sum_{\ell=1}^{\infty} \lambda^{\ell-1} T^\ell J, \qquad \lambda \in [0,1)$



Aggregation Framework



- Introduce s aggregate states, aggregation and disaggregation probs
- A composite system with both regular and aggregate states
- Two single step Bellman equations

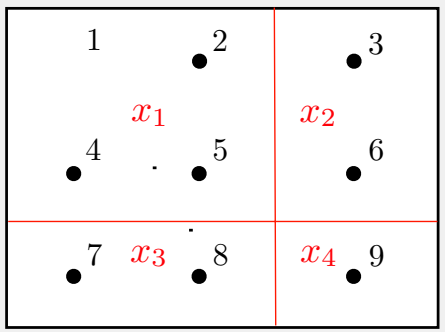
$$r = DT(\Phi r), \qquad \Phi r = \Phi DT(\Phi r)$$

  r is the cost vector of the aggregate states, Φr the cost vector of the regular states
- Natural multistep versions for bias-variance tradeoff:

$$\Phi r = \Phi DT^{(\lambda)}(\Phi r) \qquad \text{or} \qquad \Phi r = \Phi DT^{(w)}(\Phi r)$$

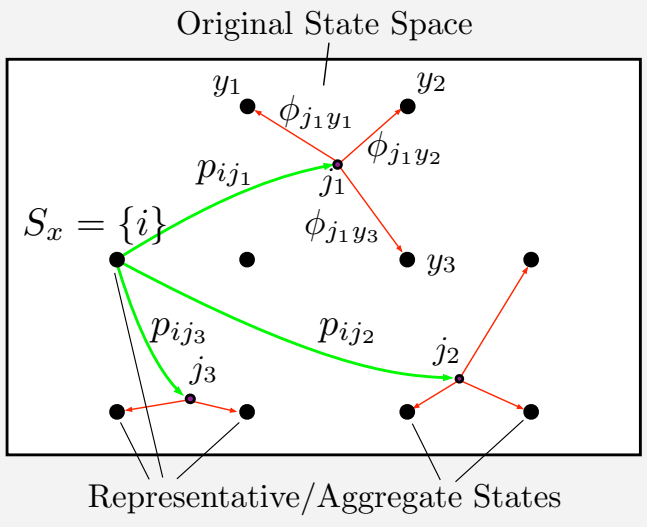
Two Common Types of Aggregation

- Hard aggregation: The aggregate states are disjoint subsets S_x of states with $\cup_x S_x = \{1, \ldots, n\}$, and $d_{xi} > 0$ only if $i \in S_x$, $\phi_{ix} = 1$ if $i \in S_x$.

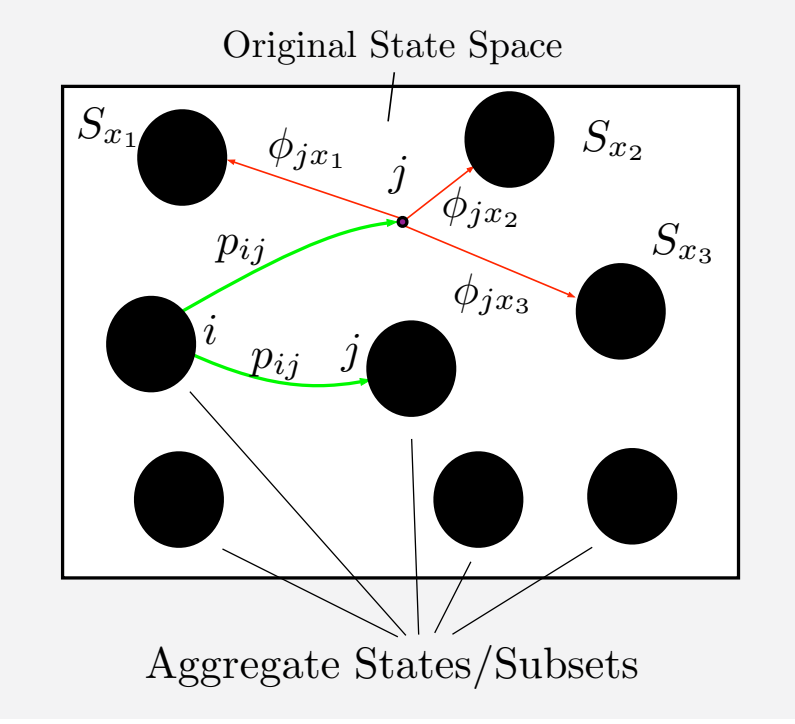


$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Aggregation with discretization grid of representative states: Each aggregate state is a single original system state $x \in \{1, \ldots, n\}$, and $d_{xx} = 1$.



A Generalization: Aggregation with Representative Features



- The aggregate states are disjoint subsets S_x of “similar" states
- Common case: S_x is a group of states with “similar features"
- Hard aggregation is a special case: $\cup_x S_x = \{1, \ldots, n\}$
- Aggregation with representative states is a special case: S_x consists of just one state

Connection with Seminorm Projection

Consider the aggregation equations

$r = DT^{(w)}(\Phi r)$, (low-dimensional) $\quad \Phi r = \Phi DT^{(w)}(\Phi r)$, (high-dimensional)

Compare them with projected equation case $\Phi r = \Pi T^{(w)}(\Phi r)$

Assume that the approximation is piecewise constant with interpolation: constant within the aggregate states, interpolated for the other states, i.e., the disaggregation and aggregation probs satisfy

$$\phi_{ix} = 1 \ \ \forall\, i \in S_x, \qquad d_{xi} > 0 \ \text{ iff } \ i \in S_x$$

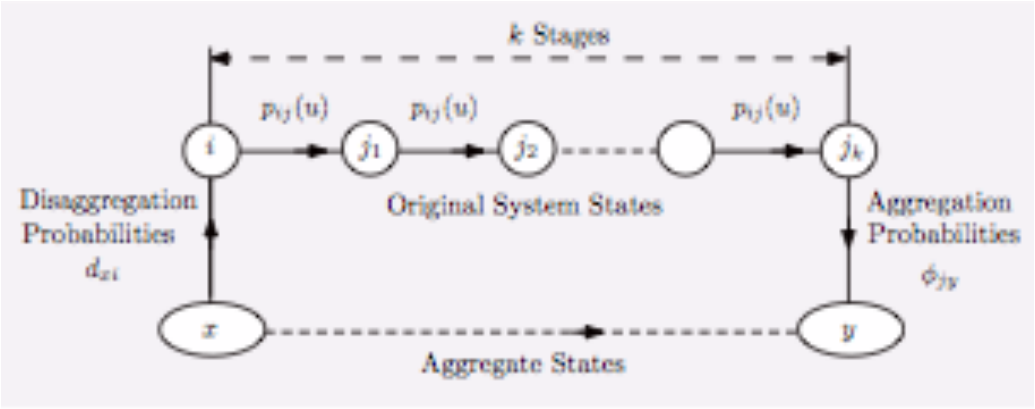
Then ΦD is a seminorm projection with

$$\xi_i = d_{xi}/s, \qquad \forall\, i \in S_x$$

This is true for the preceding aggregation schemes. Moreover, the multistep equation $\Phi r = \Phi DT^{(w)}(\Phi r)$ is a sup-norm contraction if T is.

Sampling for Aggregation

- The classic form of TD sampling does not work for multistep aggregation.
- Reason: In aggregation we need to simulate multistep cost samples involving both regular and aggregate states. This cannot be easily done with classical TD sampling.
- So we introduce a more general (free-form) sampling.
- Generate many short trajectories.



- In aggregation, the start and end states of each trajectory must be an aggregate state.
- A side benefit: A lot of flexibility for “exploration".

An Example: Projected Value Iteration for Equation $\Phi r = \Pi T^{(w)}(\Phi r)$

Exact form of projected value iteration

$$\Phi r_{k+1} = \Pi T^{(w)}(\Phi r_k)$$

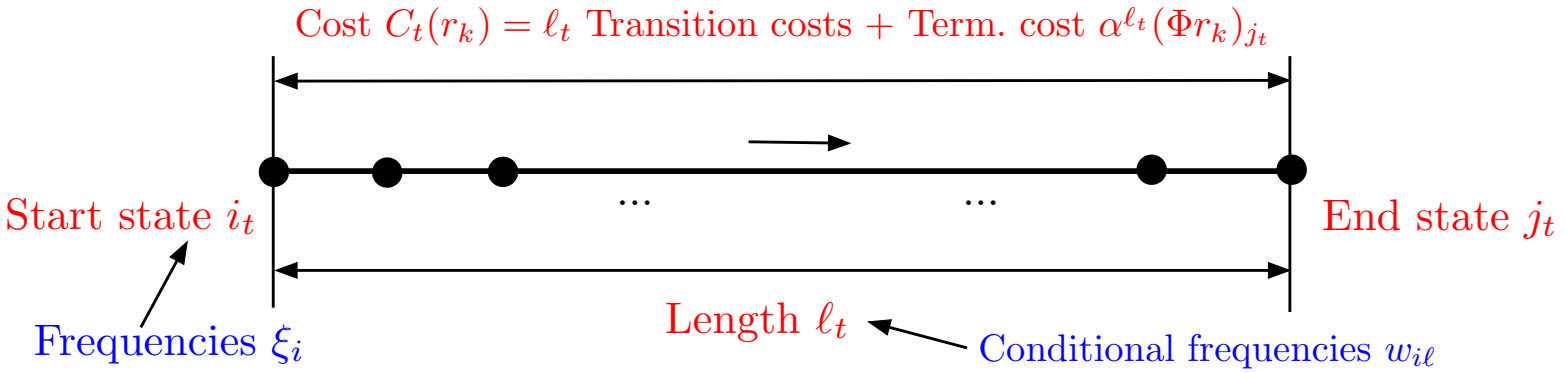
or

$$r_{k+1} = \arg\min_r \sum_{i=1}^{n} \xi_i \left(\phi(i)' r - \sum_{\ell=1}^{\infty} w_{i\ell} \big(T^{\ell}(\Phi r_k)\big)(i) \right)^2, \quad (\phi(i)': i\text{th row of } \Phi)$$

We view the expression minimized as an expected value that can be simulated with Markov chain trajectories:

- ξ_i will be the "frequency" of i as start state of the trajectories
- $w_{i\ell}$ will be the "frequency" of trajectory length ℓ when i is the start state

Simulation-Based Implementation of Projected Value Iteration



Approximation using trajectories $t = 1, \ldots, m$

$$r_{k+1} = \arg\min_r \sum_{t=1}^{m} \left(\phi(i_t)'r - C_t(r_k)\right)^2 \quad (i_t\text{: start state, } C_t(r_k)\text{: sample cost})$$

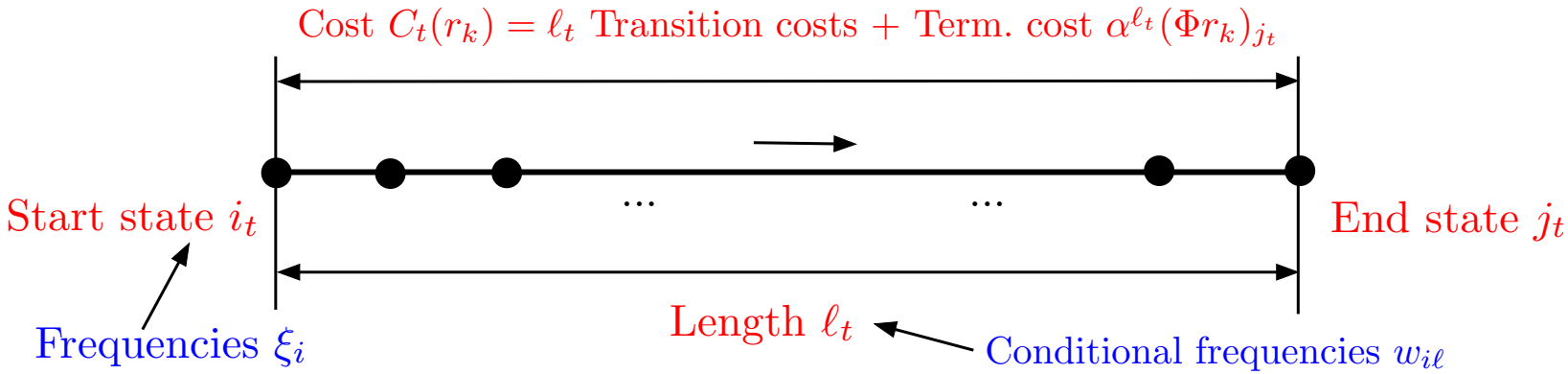
Since freq. of start state $i \to \xi_i$, freq. of start-state/length $(i, \ell) \to \xi_i w_{i\ell}$

Opt. condition for simulation-based least squares

converges to

Opt. condition for exact least squares

Matrix Inversion Method (Extension of LSTD(λ))



Find $\hat{r}$ such that

$$\hat{r} = \arg\min_r \sum_{t=1}^{m} \left(\phi(i_t)'r - C_t(\hat{r})\right)^2$$

This is a linear system of equations (the equivalent optimality condition).

Concluding Remarks

- Extension of cost function approximation methodology in DP via three interlocking ideas:
  - Seminorm projections.
  - Generalized weighted Bellman equations.
  - Free-form simulation.
- The approximation framework is general enough to include both multistep projected equations and aggregation (and other methods).
- Some of the highlights:
  - Connection between projected equations and aggregation equations.
  - Multistep aggregation methods of the TD(λ) type.
  - Use of a variety of sampling methods.
  - Flexible treatment of the bias-variance tradeoff.
- The methodology extends to the much broader field of Galerkin approximation for solving general linear equations.

Thank you!