7. Basics of Turbulent Flow

Whether a flow is laminar or turbulent depends of the relative importance of fluid friction (viscosity) and flow inertia. The ratio of inertial to viscous forces is the Reynolds number. Given the characteristic velocity scale, U, and length scale, L, for a system, the Reynolds number is $Re = UL/\nu$, where ν is the kinematic viscosity of the fluid. For most surface water systems the characteristic length scale is the basin-scale. Because this scale is typically large (1 m to 100's km), most surface water systems are turbulent. In contrast, the characteristic length scale for groundwater systems is the pore scale, which is typically quite small (< 1 mm), and groundwater flow is nearly always laminar.

The characteristic length-scale for a channel of width w and depth h is the hydraulic radius, $R_h = wh/P$, where P is the wetted perimeter. For an open channel $P = (2h + w)$ and for a closed conduit $P = 2(h+w)$. As a general rule, open channel flow is laminar if the Reynolds number defined by the hydraulic radius, $Re = UR_h/\nu$ is less than 500. As the Reynolds number increases above this limit burst of turbulent appear intermittently in the flow. As Re increases the frequency and duration of the turbulent bursts also increases until Re > O(1000), at which point the turbulence is fully persistent. If the conduit boundary is rough, the transition to fully turbulent flow can occur at lower Reynolds numbers. Alternatively, laminar conditions can persist to higher Reynolds numbers if the conduit is smooth and inlet conditions are carefully designed.

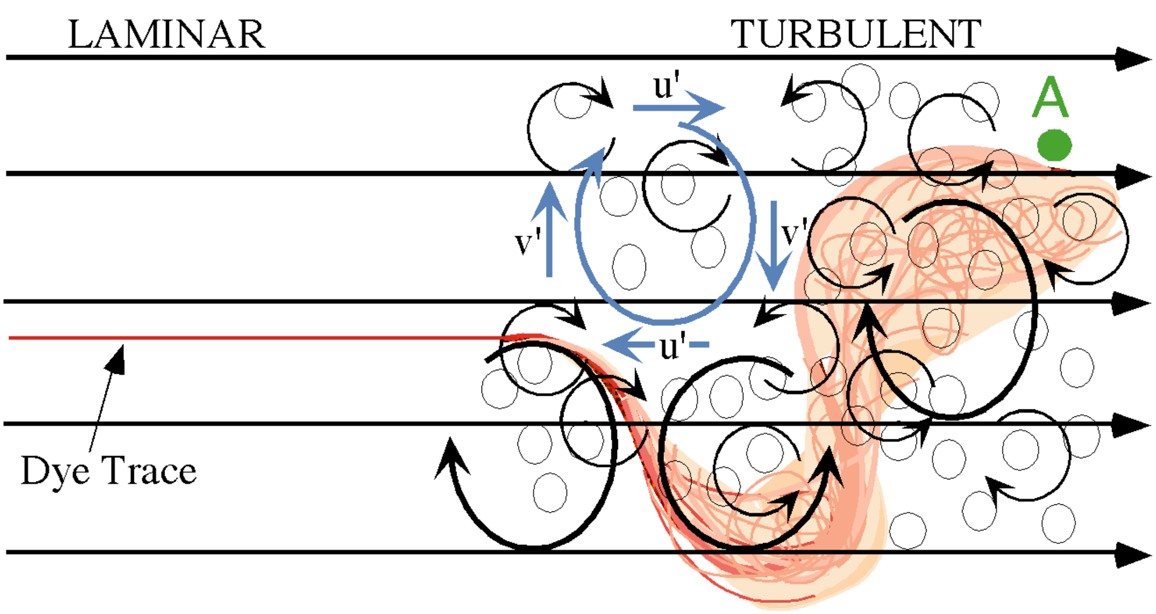


Figure 1. Tracer transport in laminar and turbulent flow. The straight, parallel black lines are streamlines, which are everywhere parallel to the mean flow. In laminar flow the fluid particles follow the streamlines exactly, as shown by the linear dye trace in the laminar region. In turbulent flow eddies of many sizes are superimposed onto the mean flow. When dye enters the turbulent region it traces a path dictated by both the mean flow (streamlines) and the eddies. Larger eddies carry the dye laterally across streamlines. Smaller eddies create smaller scale stirring that causes the dye filament to spread (diffuse).

Characterizing Turbulence:
Turbulent eddies create fluctuations in velocity. As an example, the longitudinal (u) and vertical (v) velocity measured at point A in figure 1 are shown below. Both velocities varying in time due to turbulent fluctuations. If the flow were steady and laminar then $u = \bar{u}$ and $v = \bar{v}$ for all time t, where the over-bar denotes a time average. For turbulent flow, however, the velocity record includes both a mean and a turbulent component. We decompose the flow as follows.

$$\begin{aligned} u(t) &= \underset{\phantom{\text{mean}}}{\bar{u}} + u'(t) \\ v(t) &= \underset{\text{mean}}{\bar{v}} + \underset{\text{turbulent fluctuation}}{v'(t)} \end{aligned} \tag{1}$$

This is commonly called a Reynolds' decomposition.

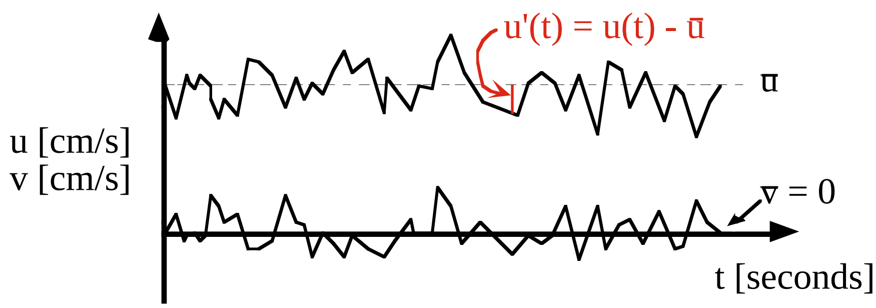


Figure 2. Velocity recorded at Point A in Figure 1.

Because the turbulent motions associated with the eddies are approximately random, we can characterize them using statistical concepts. In theory the velocity record is continuous and the mean can be evaluated through integration. However, in practice the measured velocity records are a series of discrete points, u_i. Below an overbar is used to denote a time average over the time interval t to t+T, where T is much longer than any turbulence time scale, but much shorter than the time-scale for mean flow unsteadiness, e.g. wave or tidal fluctuation.

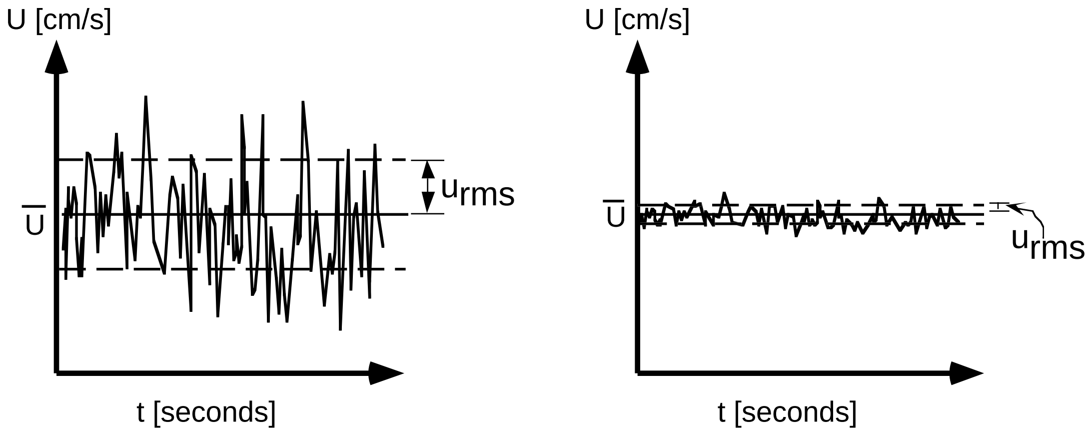
Mean velocity:
$$\bar{u} = \underbrace{\int_t^{t+T} u(t)\,dt}_{\text{continuous record}} = \underbrace{\frac{1}{N}\sum_1^N u_i}_{\text{discrete, equi-spaced pts.}} \tag{2}$$

Turbulent Fluctuation:
$$\begin{aligned} u'(t) &= u(t) - \bar{u} &&: \text{continuous record} \\ u'_i &= u_i - \bar{u} &&: \text{discrete points} \end{aligned} \tag{3}$$

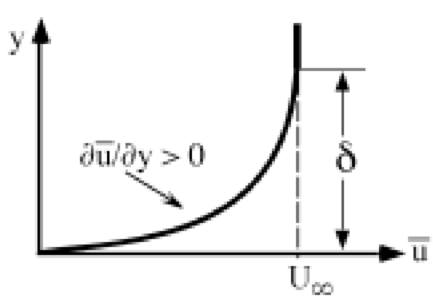
Turbulence Strength:
$$u_{rms} = \underbrace{\sqrt{\overline{u'(t)^2}}}_{\text{continuous record}} = \underbrace{\sqrt{\frac{1}{N}\sum_{i=1}^N (u'_i)^2}}_{\text{discrete, equi-spaced pts}} \tag{4}$$

Turbulence Intensity: $u_{rms}/\bar{u}$ (5)

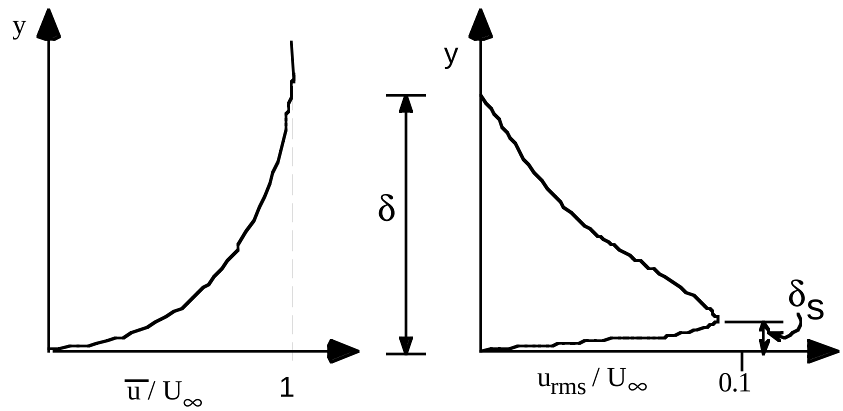
The subscript 'rms' stands for root-mean-square. You should recognize the definition of u_{rms} given in (4) as the standard deviation of the set of "random" velocity fluctuations, u_i'. Similar definitions apply to the lateral and vertical velocities, v(t) and w(t). A larger u_{rms} indicates a higher level turbulence. In the figure below, both records have the same mean velocity, but the record on the left has a higher level of turbulence.



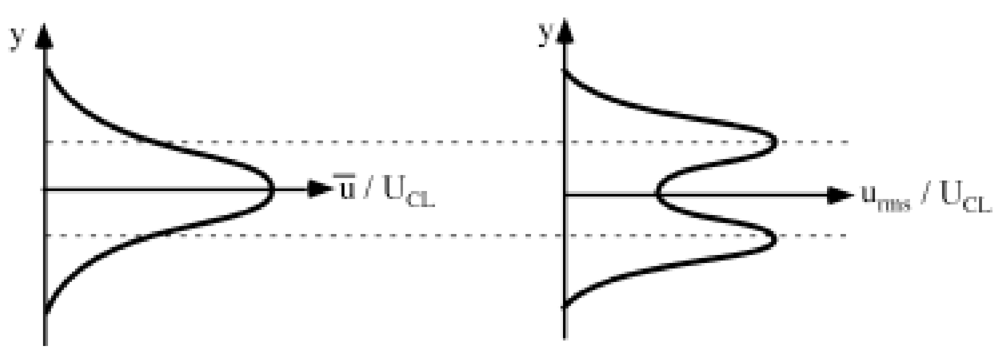
Mean Velocity Profiles - Turbulent Boundary Layers:
Near a solid boundary the flow has a distinct structure, called a boundary layer. The most important aspect of a boundary layer is that the velocity of the fluid goes to zero at the boundary. This is called the "no-slip" condition, *i.e*. the fluid velocity matches (has no slip relative to) the boundary velocity. This arises because of *viscosity*, ν, which is a fluid's resistance to flowing, *i.e*. fluid friction. The fluid literally sticks to the boundary. The higher its viscosity, the more a fluid resists flowing. Honey, for example, has a higher viscosity than water. The kinematic viscosity of water is $\nu = 0.01\ cm^2/s$. The figure below depicts a typical mean velocity profile, $\bar{u}(y)$, above a solid boundary. The vertical axis (y) denotes the distance above the boundary. The fluid velocity at the boundary (y = 0) is zero. At some distance above the boundary the velocity reaches a constant value, U_∞, called the free stream velocity. Between the bed and the free stream the velocity varies over the vertical coordinate. The spatial variation of velocity is called *shear*. The region of velocity shear near a boundary is called the momentum boundary layer. The height of the boundary layer, δ, is typically defined as the distance above the bed at which $\bar{u} = 0.99\ U_\infty$.



Shear Produces Turbulence:
Turbulence is an instability generated by shear. The stronger the shear, the stronger the turbulence. This is evident in profiles of turbulence strength (u_{rms}) within a boundary layer (see figure below). The shear in the boundary layer decreases moving away from the bed, $\partial\left(\partial\bar{u}/\partial y\right)/\partial y < 0$, and as a result the turbulence intensity also decreases. Very close to the bed, however, the turbulence intensity is diminished, reaching zero at the bed (y=0). This is because the no-slip condition applies to the turbulent velocities as well as to the mean velocity. Thus, in a thin region very close to the bed, no turbulence is present. This region is called the *laminar sub-layer*, δ_s. Note that the profiles shown below are normalized by the free-stream velocity, U_∞. This is done to emphasize the fact that the mean and turbulent profiles within a boundary layer are *self-similar* with respect to the free stream velocity, U_∞. This means that both profiles have the same shape regardless of the absolute magnitude of the external flow, U_∞. Because of this self-similarity, we have the general rule of thumb that the turbulence level increases with the free stream velocity, $u_{rms} \sim U_\infty$, where the symbol ~ is read "scales on". In addition, as the turbulence level increases, the thickness of the laminar sub-layer decreases. In general, $\delta_s \sim (1/U_\infty)$



As a second example, consider the profiles of mean and turbulent velocity measured across a jet. The profiles are self-similar when normalized by the centerline velocity, U_{CL}. The maximum turbulence level occurs at the positions of maximum shear. At the centerline the shear is zero ($\partial\bar{u}/\partial y = 0$), and the turbulence strength is diminished.



Friction [Shear] Velocity, U_*:

Physically, we know that the turbulence level scales on the shear, $u_{rms} \sim \partial \bar{u}/\partial y$. But this scale relationship is not dimensionally consistent, so we introduce a velocity scale to represent the shear strength. This velocity scale, u_*, is called the shear velocity, or the friction velocity, and it characterizes the shear at the boundary. The definition of u_* is based on the bed stress, τ_{bed}, i.e.

$$\tau_{bed} = \rho u_*^2, \tag{6}$$

where τ_{bed} is defined by the stress-strain relation,

$$\tau_{bed} = \rho \nu \left.\frac{\partial \bar{u}}{\partial y}\right|_{y=0}. \tag{7}$$

Thus,

$$u_* = \sqrt{\tau/\rho} = \sqrt{\nu \left(\partial \bar{u}/\partial y \right)\Big|_{y=0}} \tag{8}$$

The shear velocity characterizes the turbulence strength and laminar sub-layer thickness.

$$u_{rms} \sim u_* \tag{9}$$

$$\delta_s = 5 \nu / u_* \tag{10}$$

Turbulent Velocity Profile: The Logarithmic Velocity Profile:

The shape of the velocity profile within a turbulent boundary layer is well-established by theory and experiment. The profile has specific characteristics very close to the bed where viscosity controls the vertical transport of momentum, and different characteristics farther from the bed where turbulence controls the vertical transport of momentum. The region closest to the boundary is called the Laminar Sub-Layer, because within the region turbulence is suppressed by viscosity. In this region the velocity profile is defined by the stress-relation given in (7). We substitute the definition given in (6) into (7) and use the approximation $\partial u/\partial y \approx u/y$ to solve for the velocity profile.

Laminar Sub-Layer [$y < \delta_s = 5 \nu / u_*$]:

$$\bar{u}(y) = u_*^2 \, y / \nu \tag{11}$$

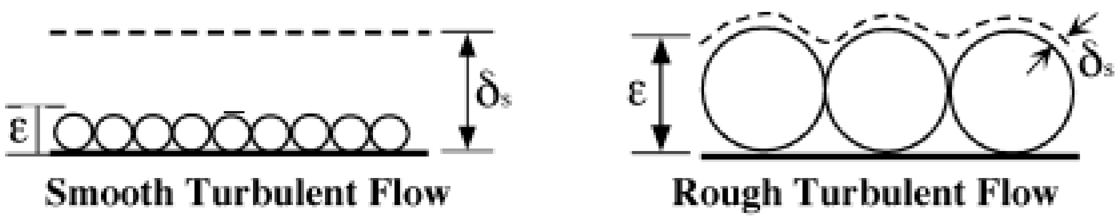
Above the Laminar Sub-Layer ($y > \delta_s$) the velocity profile is logarithmic. The profile shape depends both on the bed stress (through u_*) as well as on the bed texture, described by the characteristics roughness, y_O.

Logarithmic Layer [$y > \delta_s$]:

$$\bar{u}(y) = \frac{2.3 u_*}{\kappa} \log_{10}\left(y/y_o\right) \tag{12}$$

$\kappa = 0.4$ is an empirical constant, known as von Karman's constant.

Nikuradse studied the influence of boundary texture on velocity profile shape. He glued uniform sand grains of diameter ε, to the bed of a flume and measured the velocity profile over the bed at different flow speeds. He found two different behaviors defined by the roughness Reynolds number, $\varepsilon u_*/\nu$. For conditions with $\varepsilon u_*/\nu < 5$, yo = $\nu/9u_*$, i.e., the characteristic roughness is NOT a function of the real roughness scale. This means that the velocity profile shape, through yo, is not a function of the real roughness scale, or, simply, the logarithmic portion of the velocity profile is independent of the surface roughness under these conditions. To understand why, recall from (10) that the thickness of the laminar sub-layer, $\delta_s = 5\,\nu / u_*$. So, Nikuradse's findings simply say that when the surface texture is smaller than the laminar sub-layer ($\varepsilon < 5\,\nu / u_*$), then the flow above the laminar sub-layer does not feel the surface texture. We call this regime Smooth Turbulent Flow. When the roughness becomes larger than the laminar sub-layer, specifically $\varepsilon > (70 \text{ to } 100)\nu /u^* = 14$ to $20\,\delta_s$, then the flow above the laminar sub-layer does feel the surface texture. Under these conditions yo = $\varepsilon /30$, i.e. the characteristic roughness IS a function of the real roughness scale, and the logarithmic profile is altered, through y_o, by the surface texture. We call this regime Rough Turbulent Flow.



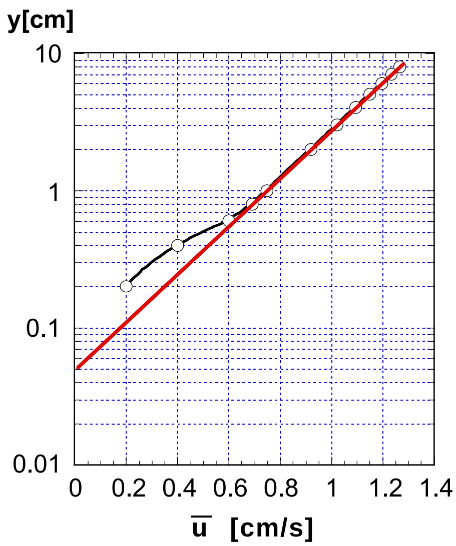
$$\frac{u_*\,\varepsilon}{\nu} < 5, \quad y_o = \frac{\nu}{9u_*} \qquad\qquad \frac{u_*\,\varepsilon}{\nu} > 70-100, \quad y_o = \frac{\varepsilon}{30}$$

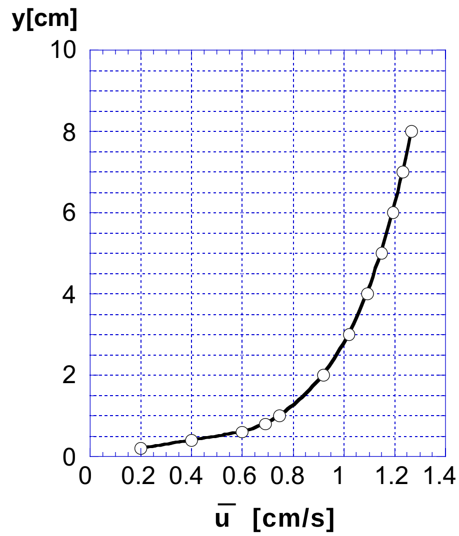
Example of Fitting a Logarithmic Profile.

An example of a mean velocity profile is graphed in two forms on the following page, using logarithmic and linear axes. The linear axes reveal the more familiar boundary layer profile. The logarithmic portion of the profile appears linear on the logarithmic axes. By fitting the logarithmic portion of the profile, we can estimate the characteristic roughness, y_o, and the friction velocity, u_*. The red line is the log-linear fit to the velocity profile. We ignore the two points closest to the bed, as these do not follow the same log-linear trend as the rest of the profile, and we suspect (and will later check) that they lie within the laminar sub-layer. From (12) the slope of the red, fitted line gives us an estimate for u_*. Specifically, we select two points on the red line, y1 = 6 cm and y2 = 0.05 cm, with velocity u1 = 1.2 cms^{-1} and u2 = 0 cms^{-1}, respectively. Then,

$$u_* = \frac{\kappa}{2.3}\frac{u2 - u1}{\log_{10}(y1) - \log_{10}(y2)} = \frac{\kappa}{2.3}\frac{1.2 - 0 \quad cm/s}{\log_{10}(6/0.05)} = 0.1\ cm/s$$

The characteristic roughness, y_o, is the y-intercept of the red line. That is, from (12), u = 0 when y = y_o. From the graph, y_o = 0.05 cm. Since $y_o > \nu/9u_* = 0.01$, the flow is not Smooth Turbulent. From (10) the laminar sub-layer thickness is 0.5 cm, which confirms that the two points closest to the boundary lie inside the laminar sub-layer. Finally, if we assume the flow is Fully Rough Turbulent, $\varepsilon = 30\, y_o = 1.5$ cm. Then the roughness Reynolds' number is $\varepsilon u_*/\nu = 15$. Because $\varepsilon u_*/\nu < 70$, we conclude that the flow is not Rough Turbulent, but in a transition between Smooth and Rough.





Turbulent Transport in the Equation of Mass Conservation

The presence of turbulence creates fluctuations in concentration. As we did with the velocity field above (see equation 1), we decompose the concentration into a temporal mean and turbulent fluctuations around that mean. As above, the over-bar indicates an average over time-scale T, which is long compared to the turbulent fluctuations.

$$C(t) = \bar{C} + C'(t) \tag{13}$$

For simplicity we start with a one-dimensional version of the equation of mass conservation (transport equation),

$$\frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} = \frac{\partial}{\partial x} D_x \frac{\partial C}{\partial x}, \tag{14}$$

into which we substitute the decomposition of velocity and concentration.

$$\frac{\partial\left(\bar{C} + C'\right)}{\partial t} + \frac{\partial\left((\bar{u} + u')(\bar{C} + C')\right)}{\partial x} = \frac{\partial}{\partial x} D_x \frac{\partial\left(\bar{C} + C'\right)}{\partial x} \tag{15}$$

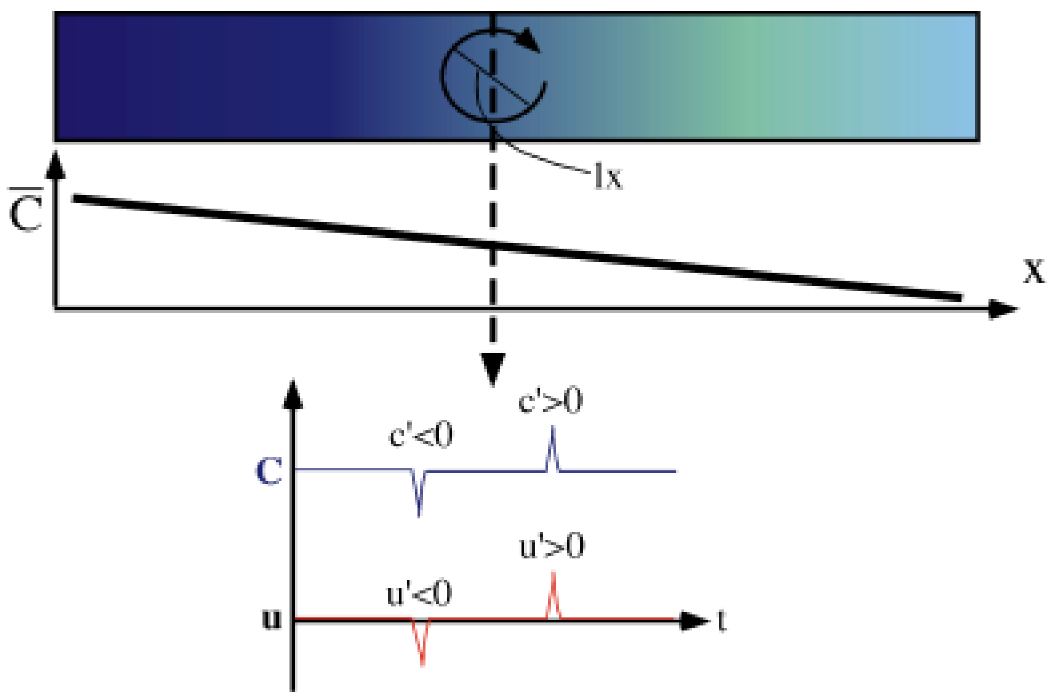
Now, we time average each term. By definition, $\overline{a'} = 0$, and $\overline{\overline{a}} = \overline{a}$. Then (15) becomes,

$$\frac{\partial \bar{C}}{\partial t} + \frac{\partial\left(\bar{u}\bar{C} + \overline{u'C'}\right)}{\partial x} = \frac{\partial}{\partial x} D_x \frac{\partial \bar{C}}{\partial x}. \tag{16}$$

The term $\overline{u'C'}$ represents the net mass flux due to turbulent advection. If we could fully calculate the turbulence field, we could calculate the turbulent flux and solve (16). Unfortunately this is quite complex and computationally intensive, and for many flows quite prohibitive. Alternatively we can devise a model for the turbulent flux in terms of the mean velocity and concentration, which are easily known. Then (16) can be readily solved. A simple mixing-length model is proposed below. It assumes the turbulent motions can be characterized by the length-scale of the eddies.

Mixing-Length Model for Turbulent Flux-

Below is a long narrow tube with linear concentration gradient $\partial \overline{C}/\partial x < 0$. There is no mean current in the tube, $\overline{u} = 0$. Consider the transport achieved by a single eddy with length-scale lx. At the top of the eddy $u' > 0$, and the eddy carries forward fluid of higher concentration, such that a probe positioned at the dashed line would momentarily record a concentration greater than the local mean when this eddy is present. That is, at the position of the dashed line, $C' > 0$ where $u' > 0$. Similarly, for this eddy, where $u' < 0$ then $C' < 0$.



The magnitude of the concentration fluctuations will be of the scale, $|C'| \sim lx\, \partial \overline{C}/\partial x$. The sign of the concentration fluctuation depends on both the sign of the concentration gradient and the sign of the velocity fluctuation. Again we consider the picture above in which $\partial C/\partial x$ is negative. The part of the eddy for which u' is also negative produces a negative c'. The part of the eddy for which u' is positive produces a positive C'. In a region with positive gradient, $\partial C/\partial x > 0$, u' positive produces C' negative, and u' negative produces C' positive. In general, when $\partial C/\partial x$ and u' have the same sign, $C' < 0$, and when $\partial C/\partial x$ have opposite sign, $C' > 0$. So, the sign of C' is -sign $(u' \partial C/\partial x)$. Using this definition for the sign of C' we can now write the turbulent advection generated by an isolated velocity fluctuation u'.

$$u' C' = - u' lx \, \partial \bar{C} / \partial x$$

Averaging over an ensemble of random fluctuations associated with the many eddies within a turbulent system, we write

$$\overline{u'C'} \sim u' C' = - u' lx \, \partial \bar{C} / \partial x. \tag{17}$$

This relation tells us that the turbulent flux behaves as a Fickian diffusion. The flux is proportional to the mean concentration gradient, and is counter gradient. Following this analogy, we define a turbulent diffusion coefficient, or turbulent diffusivity,

$$D_{t,x} \sim u' lx, \tag{18}$$

Such that the turbulent flux can explicitly modeled as an additional diffusion term,

$$\overline{u'C'} = -D_{t,x} \, \partial \bar{C} / \partial x. \tag{19}$$

Simply stated, this model shows that the turbulent flux depends on the strength of the turbulence (u') and the scale of the turbulence (lx). From our previous discussion, the strength of turbulence is characterized by the friction velocity, *i.e.* $u' \sim u_*$. In fact many length-scales of turbulence co-exist in a turbulent flow, so to apply (18) we must select the length scale that is most important to the turbulent flux. In general, this will be the largest length-scale in the system, because (18) tells us the effective diffusivity increases with eddy scale. Thus, the dominant length-scale of the turbulent transport will depend on the geometric constraints of the domain, which dictates the largest eddy scale in the domain. Returning to (16) and replacing the turbulent correlation (19), we arrive at

$$\frac{\partial \bar{C}}{\partial t} + \frac{\partial \left(\bar{u} \bar{C} \right)}{\partial x} = \frac{\partial}{\partial x} (D_x + D_{t,x}) \frac{\partial \bar{C}}{\partial x} \tag{20}$$

Thus, we have shown that the effect of turbulence on the transport equation can be modeled simply by increasing the coefficient of diffusion by an amount dictated by the strength and intensity of the turbulence. In general the turbulent diffusivity, $D_{t,x}$ is much greater than its molecular counterpart, such that the latter is simply ignored. Now, the solutions already devised for the transport equation can be applied in turbulent flow, but with the molecular diffusivity replaced by its turbulent cousin.

Finally, through similar reasoning on can quickly show that the turbulent diffusivity in the vertical and lateral dimension will scale as,

$$\begin{aligned} D_{t,y} &\sim v' ly \\ D_{t,z} &\sim w' lz. \end{aligned} \tag{21}$$

Because turbulence is often anisotropic in both length-scale ($lx \neq ly \neq lz$) and intensity ($u' \neq v' \neq w'$, we expect that the turbulent diffusivity will also be anisotropic ($D_{t,x} \neq D_{t,y} \neq D_{t,z}$).

Turbulent Diffusivity in a Channel

The turbulence in a channel may be characterized by the bed friction velocity, u_*, and a characteristic length-scale. A reasonable and convenient choice for the length-scale is the flow depth, h. Then, from (18), the turbulent diffusivity in the river scale as $D_t \sim u_* h$. The scaling constant must be determined from tracer studies. Table 1 presents range of scale coefficients compiled from multiple tracer studies.

Table 1. Turbulent Diffusivity in Channels

(x,y,z)=(streamwise, lateral, vertical)

$$D_{t,x} = (0.3 - 0.45)\, u_* h$$

$$D_{t,y} = \begin{cases} 0.15\, u_* h & \text{straight channel} \\ 0.6\, u_* h & \text{gentle meanders} \\ 3.4\, u_* h & \text{strong meanders} \end{cases}$$

$$D_{t,z} = \frac{1}{15} u_* h$$

•• Constants are empirical, except for $D_{t,z}$ which is based on theoretical models of length-scale.

The aspect ratio of most channels is such that lx >> ly > lz, so that we expect from (18) and (21) that $D_X > D_Y > D_Z$. This is indeed the case for straight channels. However, curvature (or meanders) in a channel introduce secondary circulations (lateral currents) that increase lateral mixing. With strong meanders the effect is sufficiently strong to make $D_{t,y} >> D_{t,x}$. Typical river values are: $D_{t,x}$=10 - 8000 cm^2/s, $D_{t,y}$=10 – 10000 cm^2/s, $D_{t,z}$=1 – 3000 cm^2/s.

From Table 1, to estimate the diffusivity in a river we need the channel depth and friction velocity. It is often difficult to measure sufficiently detailed velocity profiles from which to estimate the friction velocity, such that a simpler method has become common. Consider steady flow in a wide channel of width b, depth h << b, and bed-slope, $S = \sin\theta$, where θ is the angle to the horizontal. The momentum balance is between the component of fluid weight directed along the channel and the bed stress. Consider a length of channel dx. Neglecting the wall-stress (reasonable for a wide channel),

$$\rho\, g\, dx\, h\, b\, S - \tau_{bed}\, dx\, b = 0. \tag{22}$$

Using the definition of friction velocity, $\tau_{bed} = \rho u_*^2$, (21) can be solved for u_*.

$$u_* = \sqrt{ghS} \tag{23}$$