Shortest Paths in Graphs of Convex Sets

and their applications

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Wonderful coauthors

Tobia Marcucci

Jack Umenberger

Russ Tedrake

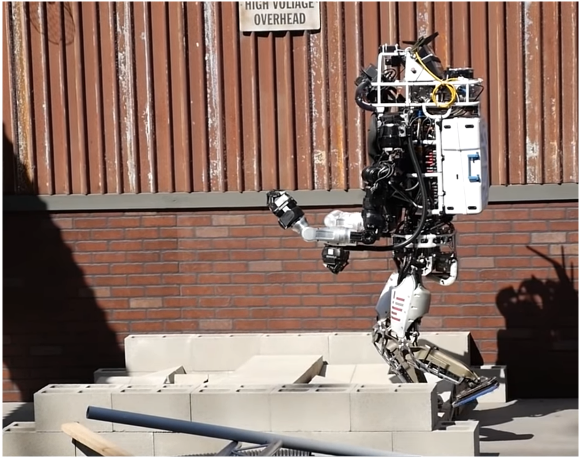
References

- Marcucci, Umenberger, Parrilo, Tedrake "Shortest paths in graphs of convex sets," arXiv:2101.11565

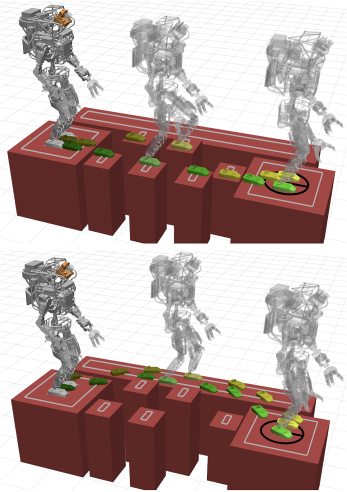
Outline

- Motivation (for us): robotics and planning
- Graphs of Convex Sets framework
- Shortest paths on GCS
- Convex relaxation and mixed-integer formulation
- Numerical results and implementation
- Conclusions and future directions

Footstep planning for legged robots

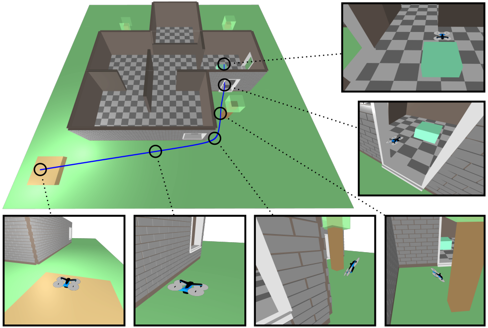


[MIT DRC (DARPA Robotics Challenge) Team]



[Deits and Tedrake '14]

Motion planning



Drone navigation in 3D environments

[1] Marcucci, Petersen, von Wrangel, Tedrake "Motion planning around obstacles with convex optimization" (2022)

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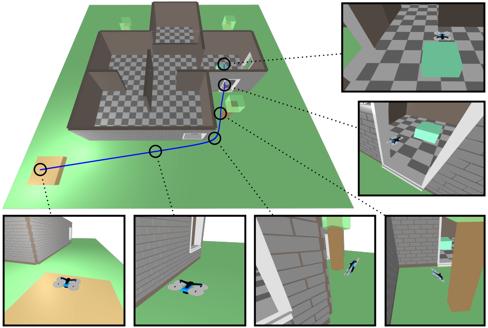

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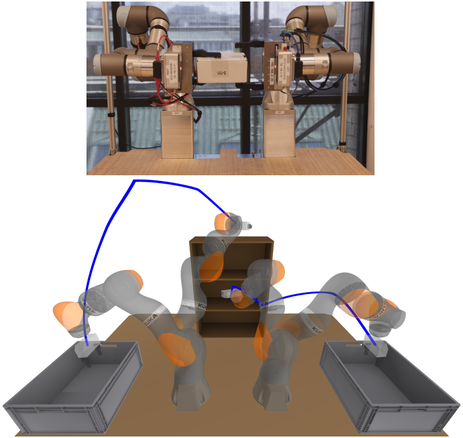


Manipulation with multiple robots/tasks

[1] Marcucci, Petersen, von Wrangel, Tedrake "Motion planning around obstacles with convex optimization" (2022)

State of the art in robot motion planning

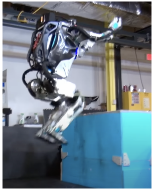
Atlas motion planning [1,2]

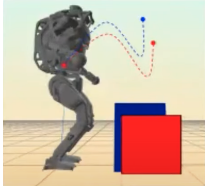
Offline:

- Library of template behaviors

Online:

- Continuous blending of behaviors
- Parametric convex optimization





[1] Kuindersma "Recent progress on Atlas, the world's most dynamic humanoid robot" (MIT Robotics Today 2020)

[2] Deits "Making Atlas Dance, Run, and Jump" (6th Workshop on Legged Robots, ICRA 2022)

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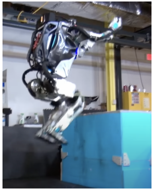

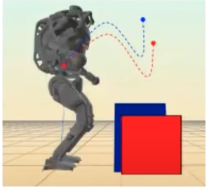

- The discrete sequence of behaviors is hand-designed
- The robot can't react to structural changes in the environment

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Motivation

- Interesting, challenging, and practically relevant problems
- Both continuous and combinatorial features
- Moderately high dimension ($10^1 - 10^4$ variables)
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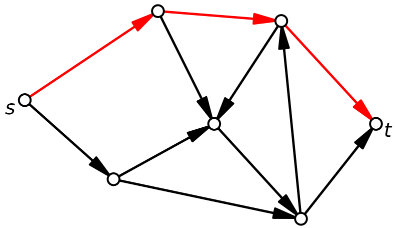
Our proposal: *Shortest paths in Graphs of Convex Sets (GCS)*

Big picture

Shortest-Path Problem (SPP)



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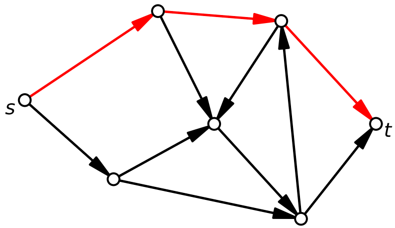

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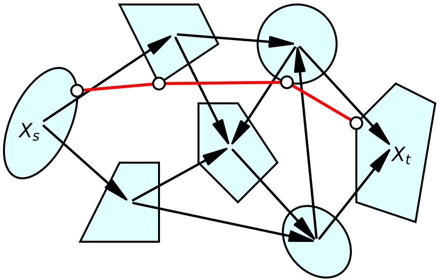

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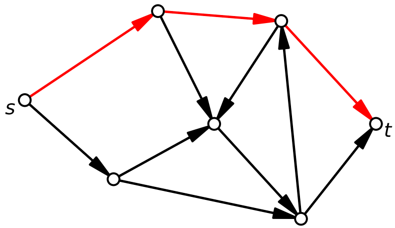

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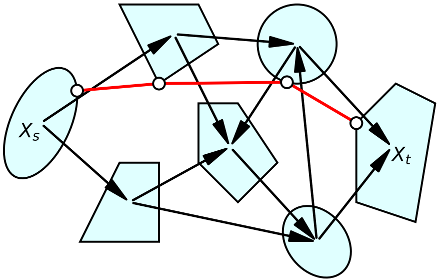

- Very versatile problem formulation

Big picture

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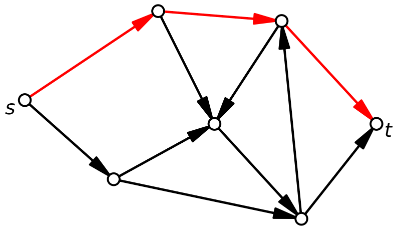

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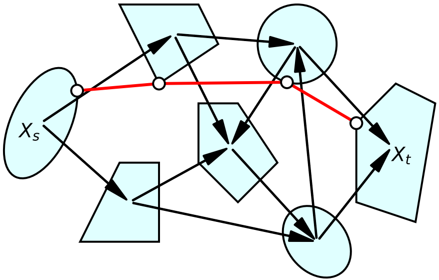

- Very versatile problem formulation
- Efficiently solvable in practice (although NP-hard)
  - Mixed-integer formulation with very tight convex relaxation

GCS gracefully blends discrete and continuous

Why? Highlights interactions and feedback between:

Structure / Whole

- High-level decisions
- Discrete "combinatorial skeleton" of trajectory/motion (e.g. obstacle avoidance)

Details / Parts

- Lower-level optimization considerations (fuel, time, cost, etc)
- Typically continuous and "nice".

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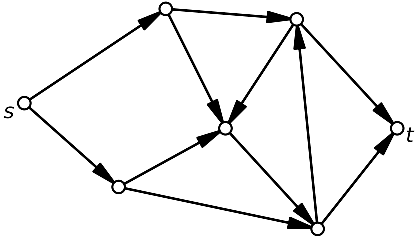
Useful guiding principle

"Easy" problems should remain easy.

SPP in a graph of convex sets

- Directed Graph $G = (V, E)$



Joint optimization over paths AND vertex locations!

SPP in a graph of convex sets

- Directed Graph $G = (V, E)$
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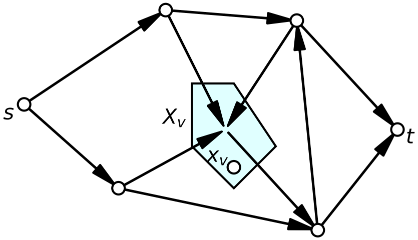

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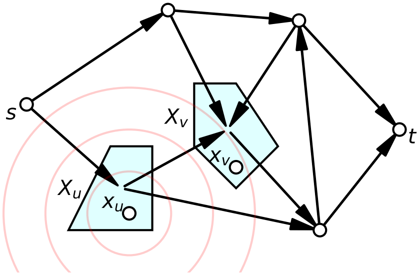
$$\ell_e : X_u \times X_v \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

  - Can enforce convex constraints $(x_u, x_v) \in X_e$



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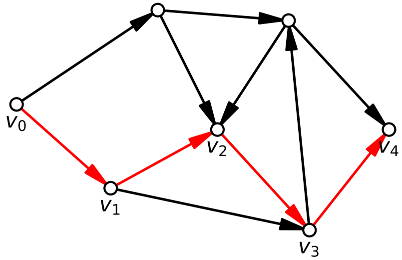
- A path π is a
  - Sequence of distinct vertices $(v_k)_{k=0}^{K}$
  - $v_0 = s$ and $v_K = t$
  - $(v_k, v_{k+1}) \in E$



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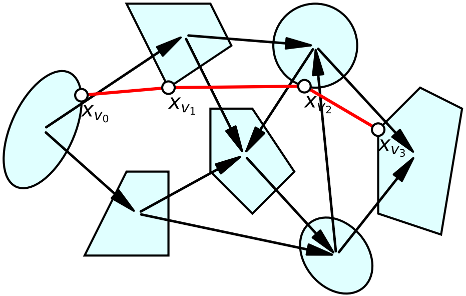


Shortest-path problem (GCS)

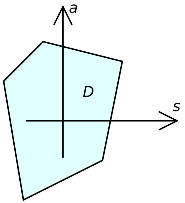
$$\min_{\pi \in \Pi} \min_{x \in X} \sum_{e \in E_\pi} \ell_e(x_u, x_v)$$

Joint optimization over paths AND vertex locations!

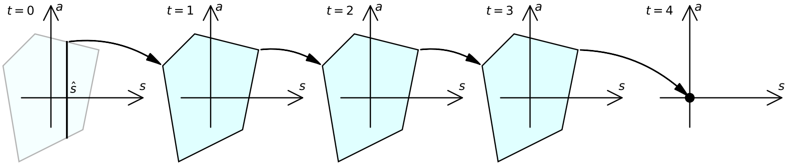
Optimal control as an SPP

Constrained linear regulation

$$\begin{array}{lll} \text{minimize} & \sum_{t=0}^{T-1} c(s_t, a_t) & \\ \text{subject to} & s_{t+1} = As_t + Ba_t, & \forall t \\ & (s_t, a_t) \in D, & \forall t \\ & s_0 = \hat{s}, \ s_T = 0 & \end{array}$$



Shortest-path problem



- Edge lengths

  $$c(s_t, a_t)$$

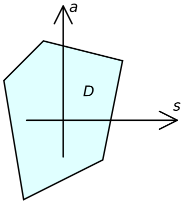
- Edge constraints

  $$s_{t+1} = As_t + Ba_t$$

Optimal control as an SPP

Minimum time

$$\begin{aligned} \text{minimize} \quad & T \\ \text{subject to} \quad & s_{t+1} = As_t + Ba_t, \qquad \forall t \\ & (s_t, a_t) \in D, \qquad \forall t \\ & s_0 = \hat{s}, \;\; s_T = 0 \end{aligned}$$



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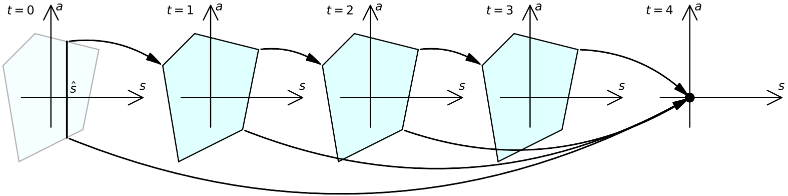


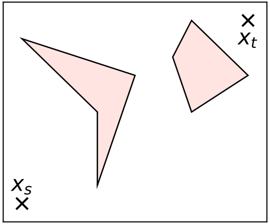
- Edge lengths

  $$1$$

- Edge constraints

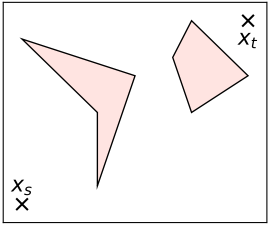
  $$s_{t+1} = As_t + Ba_t$$

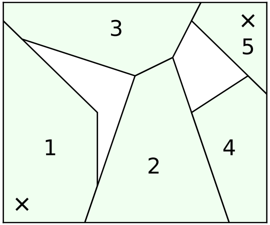
Collision-free motion planning as an SPP



Connect x_s to x_t via a collision-free polygonal line

Collision-free motion planning as an SPP





Connect x_s to x_t via a collision-free polygonal line

- Decompose free space in safe convex regions

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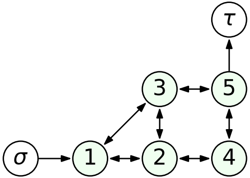
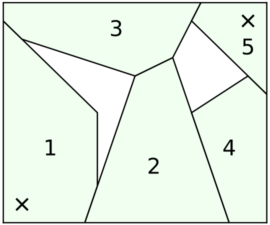
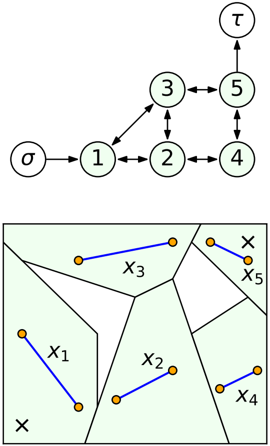




Connect x_s to x_t via a collision-free polygonal line

- Decompose free space in safe convex regions
- Construct adjacency graph

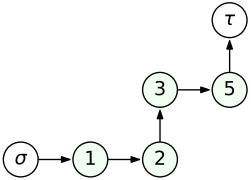
Collision-free motion planning as an SPP

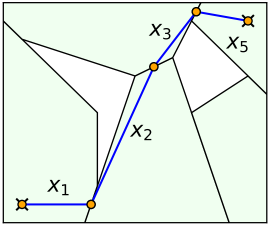


Connect x_s to x_t via a collision-free polygonal line

- Decompose free space in safe convex regions
- Construct adjacency graph
- Assign a line segment to each region
  - Segment $\in$ region $\Leftrightarrow$ start and end point $\in$ region

Collision-free motion planning as an SPP





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- Decompose free space in safe convex regions
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- Continuity enforced as an edge constraint

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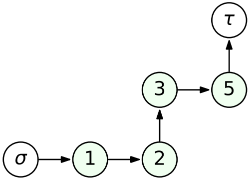

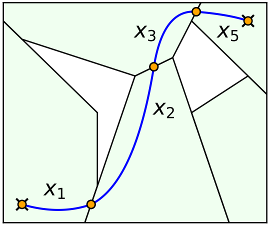

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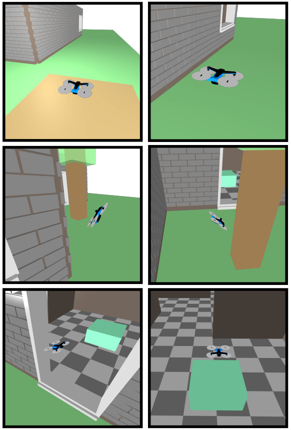
- Extends to polynomials using Bézier curves

Collision-free motion planning as an SPP



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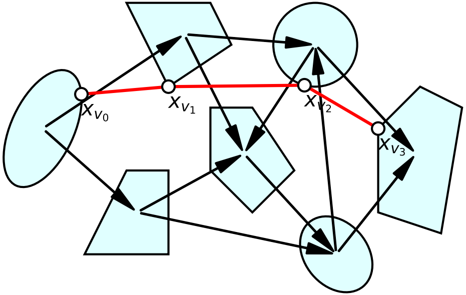
- Extends to polynomials using Bézier curves
- Takes into account the timing of the trajectory
  - Velocity constraints
  - Continuity of derivatives
  - Trajectory-duration constraints

Complexity?

- Fixing *either* π or x, problem is "easy":
  - Fixed π (sequence): convex optimization problem over x_v
  - Fixed x_v (locations): "standard" shortest path problem
- Unfortunately, NP-hard if we search for both

Bad news, but certainly expected...



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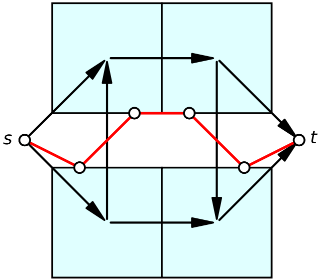
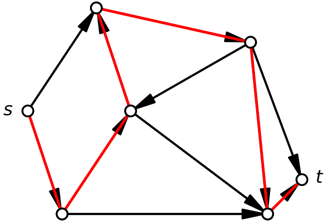
NP-Hardness: reduction from Hamiltonian-path problem

$$\ell_e(x_u, x_v) = \|x_v - x_u\|_2^2$$



"Is there a path that visits each vertex?"

Hamiltonian-path problem



- One of Karp's 21 NP-complete problems
- Reducible to our SPP in polynomial time
  - $\Rightarrow$ SPP/GCS is NP-hard

(Some) Related work

Location/allocation problems (Cooper 1963)

Fermat-Weber, facility location, etc.

Computational geometry

Zookeeper problem, watchman problems, safari problems, art gallery problems, etc.
Typically only 2-3 dimensions, approximation algorithms.

Graph problems with neighborhoods

Variants of the TSP and the MSTP, sometimes with quite strong restrictions on the neighborhoods (e.g., polygonal regions on the 2D plane). Very low dimensional, tackled using mixed-integer nonconvex programming [1,2,3] – not competitive with our approach

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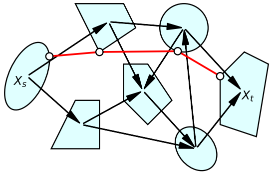
[1] Gentilini et al. "The travelling salesman problem with neighbourhoods: MINLP solution" (Optimization Methods and Software 2013)
[2] Blanco et al. "Minimum spanning trees with neighborhoods: Mathematical programming formulations and solution methods" (Eur. Jour. of OR 2017)
[3] Burdick et al. "From multi-target sensory coverage to complete sensory coverage" (ICRA 2021)

Our solution approach



- A compact mixed-integer convex program
  - With very tight convex relaxation
  - Quite often exact, otherwise do rounding or B&B

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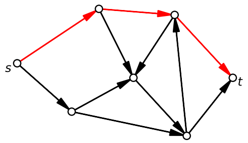

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Construction of the mixed-integer program

1. Integer program of classical SPP
   - Path parameterized using a "flow" variable $y_e \in \{0,1\}$ per edge e
   - Linear constraints enforce conservation of flow

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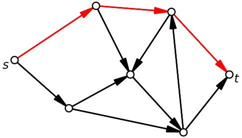

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Construction of the mixed-integer program

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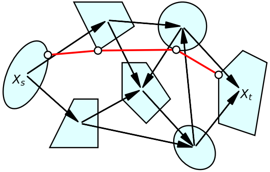

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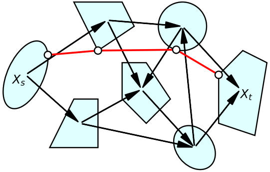
Construction of the mixed-integer program

1. Linear program of classical SPP
   - Path parameterized using a "flow" variable $y_e \in [0, 1]$ per edge e
   - Linear constraints enforce conservation of flow
2. Natural extension to graph of convex sets
   - Yields bilinear program (products between vertex positions x_v and flows y_e)

Our solution approach



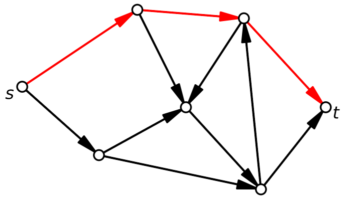
- A compact mixed-integer convex program
  - With very tight convex relaxation
  - Quite often exact, otherwise do rounding or B&B

Construction of the mixed-integer program

1. Linear program of classical SPP
   - Path parameterized using a "flow" variable $y_e \in [0, 1]$ per edge e
   - Linear constraints enforce conservation of flow
2. Natural extension to graph of convex sets
   - Yields bilinear program (products between vertex positions x_v and flows y_e)
3. Set-based perspective convex relaxation of the bilinearities
   - Exact when $y_e \in \{0, 1\}$ for all edges e

Step 1: Linear program of classical SPP

- Convex hull of paths well understood (flow polytope)
- Edge costs c_e are nonnegative scalars
- Path parameterized by the flows $\varphi_e \in [0, 1]$



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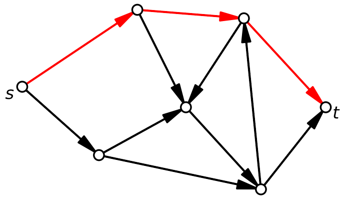
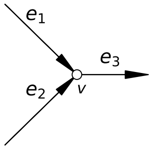


Min-cost flow LP

$$\begin{array}{ll} \text{minimize} & \sum_{e \in E} c_e \varphi_e \\ \text{subject to} & \varphi_v \in \Phi_v, \qquad \forall v \in V \end{array}$$

- φ_v is vector of flows incident with v
- Φ_v is (local) flow polytope
  - Flows are nonnegative
  - Flow through a vertex is conserved and at most one



$\varphi_v = (\varphi_{e_1}, \varphi_{e_2}, \varphi_{e_3})$

Step 2: Bilinear program

Straightforward extension of the LP

$$\begin{array}{rll} \text{minimize} & \displaystyle\sum_{e=(u,v)\in E} \ell_e(x_u, x_v)\varphi_e & \\ \text{subject to} & \varphi_v \in \Phi_v, \; x_v \in X_v, & \forall v \in V \end{array}$$

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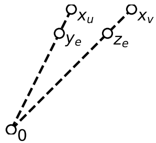
Introduce auxiliary variables y_e and z_e

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{e\in E} \ell_e\left(\frac{y_e}{\varphi_e}, \frac{z_e}{\varphi_e}\right)\varphi_e & \\ \text{subject to} & \varphi_v \in \Phi_v,\ x_v \in X_v, & \forall v \in V \\ & y_e = \varphi_e x_u,\ z_e = \varphi_e x_v, & \forall e = (u, v) \in E \end{array}$$



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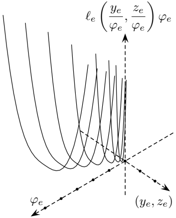
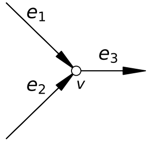
- Convex objective (perspective function of ℓ_e)
- Nonconvex constraints (bilinear)



[Combettes, '16]

Step 2: Compacting the notation

- Each edge has two "spatial" variables y_e, z_e (origin/destination)
- Bilinear constraints $y_e = \varphi_e x_u$ and $z_e = \varphi_e x_v$ for all edges $e = (u, v)$
- At each node, organize variables in matrices



$$M_v = \begin{bmatrix} z_{e_1} & z_{e_2} & y_{e_3} \end{bmatrix} = \begin{bmatrix} \varphi_{e_1} x_v & \varphi_{e_2} x_v & \varphi_{e_3} x_v \end{bmatrix} = x_v \varphi_v^\top$$

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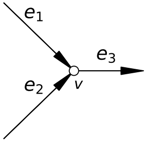

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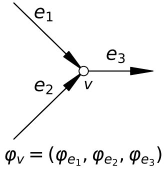

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The bilinear program

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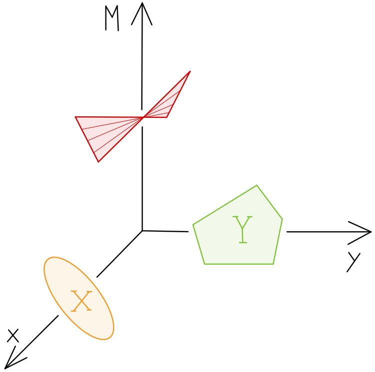
Set-based convex relaxation of the bilinearities (I)

Goal: find (good) convex outer approximation Ω' of

$$\Omega = \left\{(x, y, M) : x \in X,\ y \in Y,\ M = xy^\top\right\}$$



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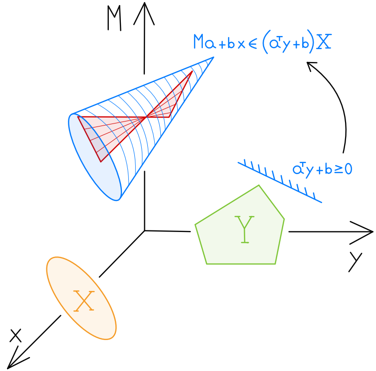

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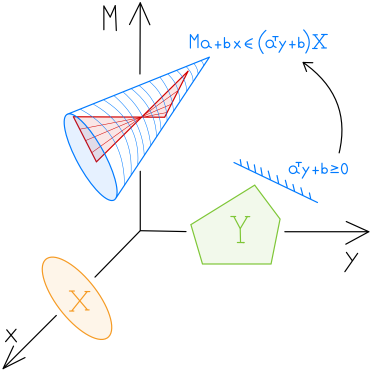

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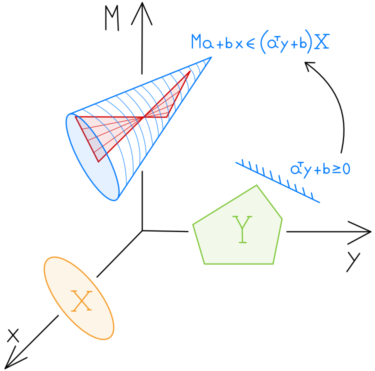

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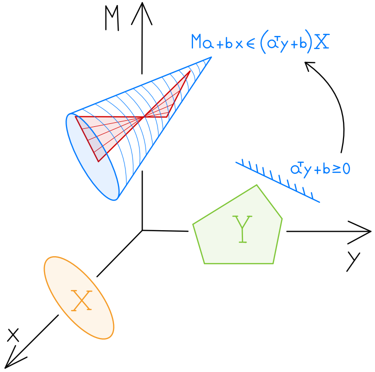

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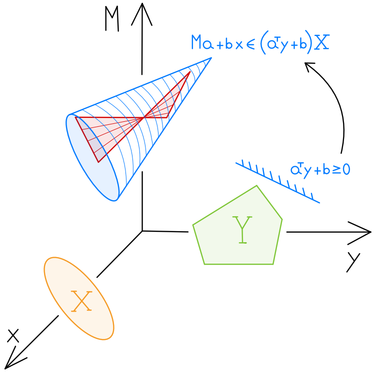
- Constraint is convex in (x, y, M) !
- Set-based relaxation, don't care about *description* of X (cf. Lovasz-Schrijver vs. RLT, also OOP-friendly)

Set-based convex relaxation of the bilinearities (II)

Goal: Find (good) convex outer approximation Ω' of

$$\Omega = \left\{(x, y, M) : x \in X,\ y \in Y,\ M = xy^\top\right\}$$

For simplicity, let Y be polyhedral (e.g., flow polytope).



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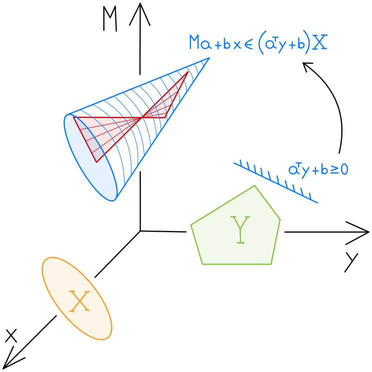

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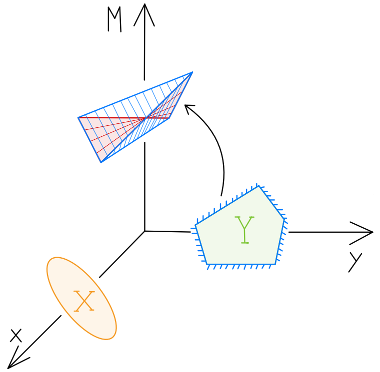

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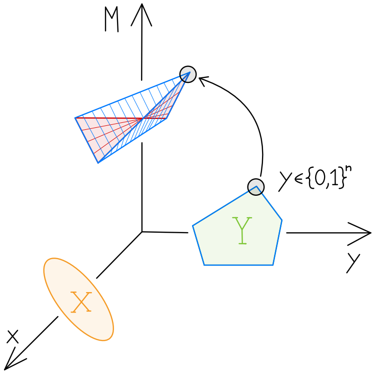

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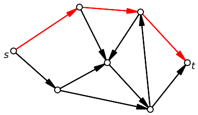
- (persistence) $\Omega' = \Omega$ when restricting $y \in \{0, 1\}^n$

Shortest path: standard vs GCS



Problem statement

$$\min \sum_{e \in E_\pi} c_e \quad \text{s.t.} \quad \pi \in \Pi$$

(Mixed Integer) Convex program

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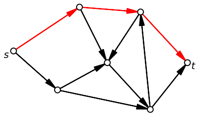

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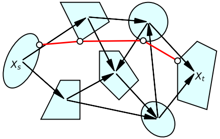

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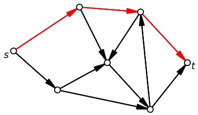

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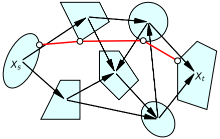

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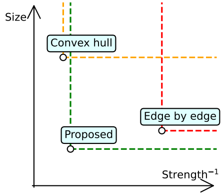
Further comments

Convex relaxation

- Applies to sets $\{(\alpha, \beta, M) : \alpha \in A,\ \beta \in B,\ M = xy^\top\}$
- Very tight in practice
  - Exact when sets X_v are singletons
- Compact: $O((|V| + |E|)\dim(X_v))$ variables and constraints
- Typically, linear or second-order-cone programs



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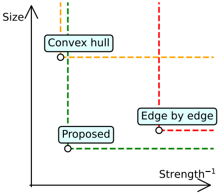

What if (after solving) flows are not 0/1?

A. Rounding the solution of the convex relaxation

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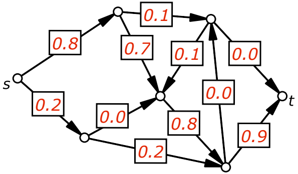

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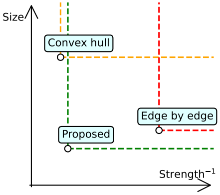

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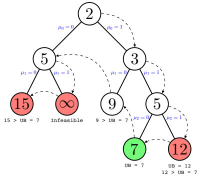
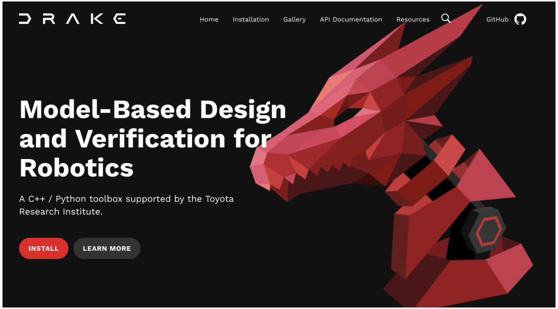
B. Branch and bound



Mature software implementation in Drake

- Implemented in robotics software Drake (TRI), C++/Python
- Off-the-shelf solvers (Gurobi/MOSEK/SCS/CSDP/ SNOPT/IPOPT/NLOPT)
- Customized solvers/algorithms (experimental)



Give it a try: `pip install drake`

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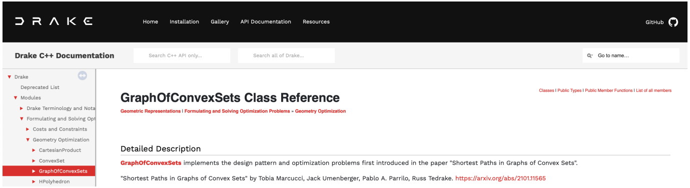
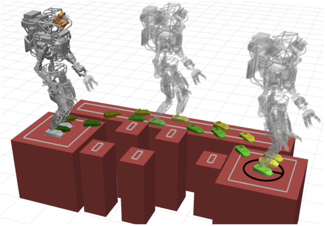
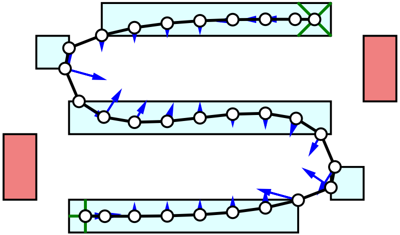
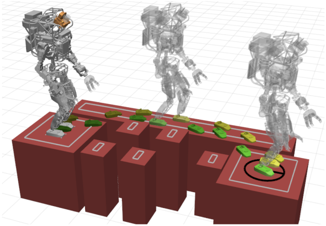


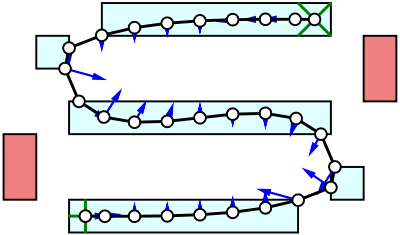
Optimal control of a hybrid system



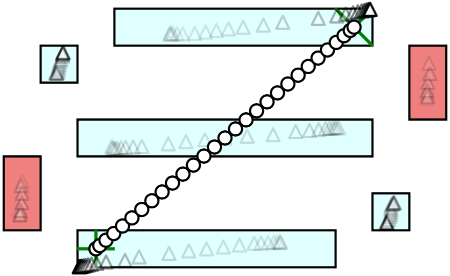


Optimal control of a hybrid system





[Moehle and Boyd '15], [Marcucci and Tedrake '19]



- $\dfrac{\text{Cost relaxation}}{\text{Cost MICP}} = 7\%$
- Branch-and-bound time 218s

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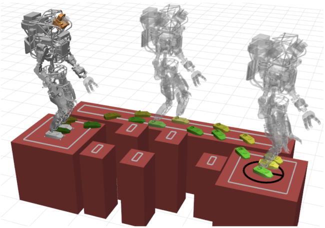
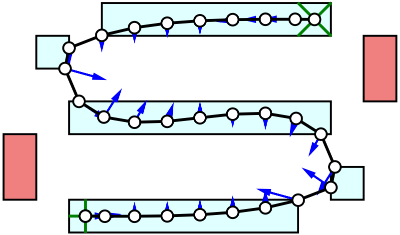
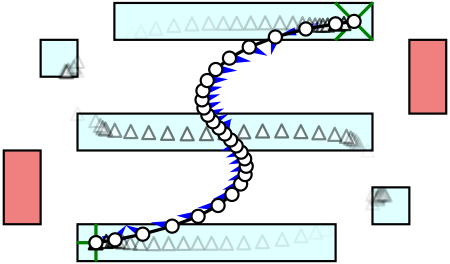


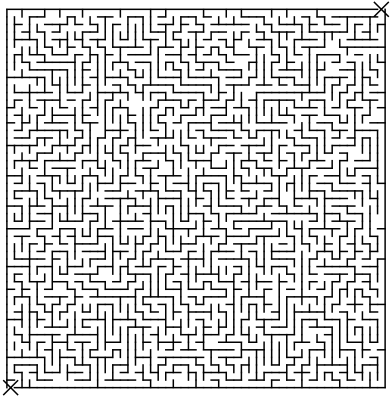


Shortest-path formulation



- $\dfrac{\text{Cost relaxation}}{\text{Cost MICP}} = 80\%$
- Branch-and-bound time 1.3s

Motion planning in a maze



- Easy to find a path via discrete graph search
- If we have differential costs and constraints?
  - Local optimization: hopeless
  - Sampling based: inefficient and non-differentiable
  - Prev. mixed-integer: $(\#\text{ cells})^2 \approx 6 \cdot 10^6$ binaries

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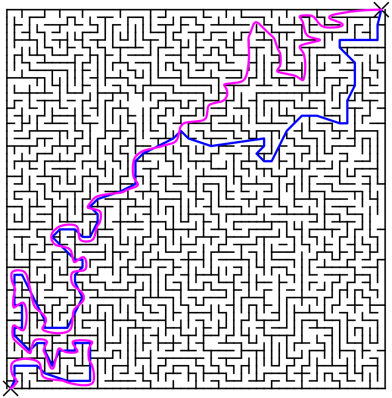

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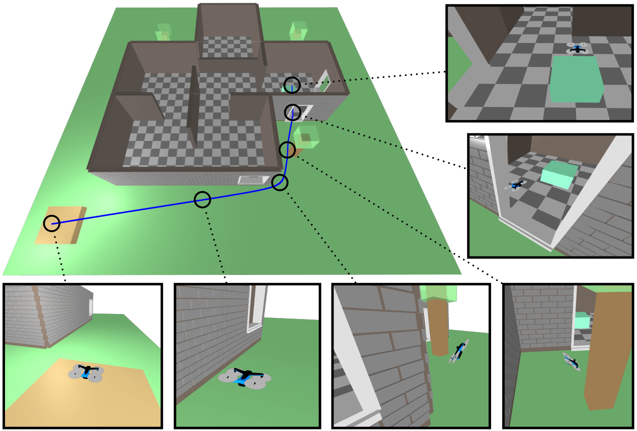
Graphs of convex sets

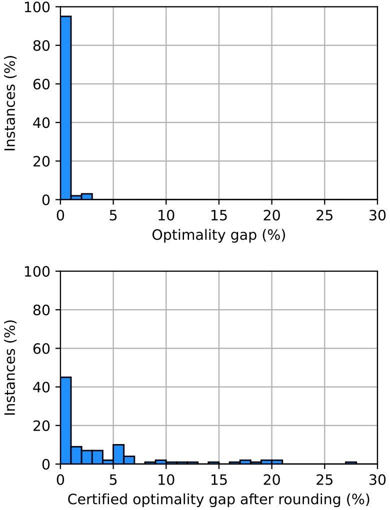
- Minimum distance
- Minimum time
  - With velocity limits and acceleration penalty
- Convex relaxation is exact in both cases!
- Only $O(\# \text{ cells})$ flow variables

Quadrotor flying around obstacles

- Exact decomposition of free space in convex sets
- Planning in (x, y, z) + differential flatness
- Penalties on length, velocity, acceleration, and duration

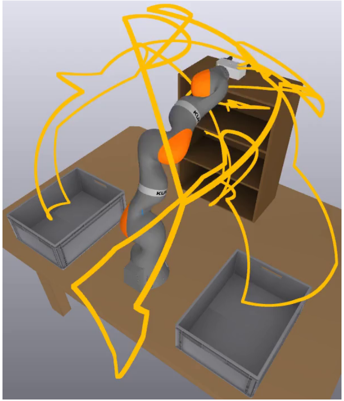
Convex relaxation + randomized rounding



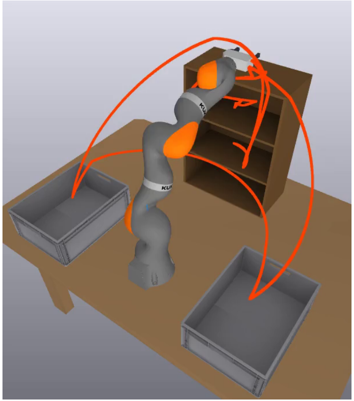


Comparison with Probabilistic RoadMap (PRM)

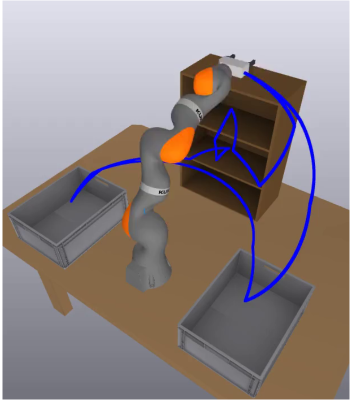
PRM



PRM with shortcuts



Graph of Convex Sets (GCS)



- Convex relaxation + rounding
- [Amice et al., '22] for the decomposition of configuration space

Comparison with Probabilistic RoadMap (PRM)

PRM	PRM with shortcuts	Graph of Convex Sets (GCS)

- Convex relaxation + rounding
- [Amice et al., '22] for the decomposition of configuration space

Planning in 14 dimensions

- Collision-free motion planning in 14 dimensions using convex optimization
- PRM hardly scales beyond 7/8 dimensions

Preliminary hardware results

Motion generated via a single convex optimization!

Wrapping up

Shortest-path problem in graphs of convex sets

- Exciting new optimization framework, flexible and powerful
- Efficiently solvable in practice
  - Tight convex relaxation + rounding
  - Strong mixed-integer convex formulation + branch and bound

Current and future directions

- Customized ADMM solver on GPU (eventually a standalone toolbox)
- Alternative algorithmic approaches? Scale to huge graphs?
- Other combinatorial problems in graphs of convex sets (TSP, MSTP)
- Extensions/applications: underactuated dynamics, temporal logic, stochastics, SLAM...

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- Your new algorithms?

Thanks for your attention!

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Questions?