

CONTINUOUS-TIME AVERAGE-PRESERVING OPINION DYNAMICS WITH OPINION-DEPENDENT COMMUNICATIONS*

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Abstract. We study a simple continuous-time multiagent system related to Krause’s model of opinion dynamics: each agent holds a real value, and this value is continuously attracted by every other value differing from it by less than 1, with an intensity proportional to the difference. We prove convergence to a set of clusters, with the agents in each cluster sharing a common value, and provide a lower bound on the distance between clusters at a stable equilibrium, under a suitable notion of multiagent system stability. To better understand the behavior of the system for a large number of agents, we introduce a variant involving a continuum of agents. We prove, under some conditions, the existence of a solution to the system dynamics, convergence to clusters, and a nontrivial lower bound on the distance between clusters. Finally, we establish that the continuum model accurately represents the asymptotic behavior of a system with a finite but large number of agents.

Key words. multiagent systems, consensus, opinion dynamics

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1. Introduction. We study a continuous-time multiagent model: each of n agents, labeled $1, \dots, n$, maintains a real number (“opinion”) $x_i(t)$, which is a continuous function of time and evolves according to the integral equation version of

$$(1.1) \quad \dot{x}_i(t) = \sum_{j: |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t)).$$

This model has an interpretation in terms of opinion dynamics: an agent considers another agent to be a neighbor if their opinions differ by less than 1, and agent opinions are continuously attracted by their neighbors’ opinions. Numerical simulations show that the system converges to clusters inside which all agents share a common value. Different clusters lie at a distance of at least 1 from each other, and often approximately 2. This is illustrated in Figure 1.1, which represents the time evolution of the values $x_i(t)$ of 1000 agents with randomly and uniformly distributed initial values. Similar behaviors are observed for other distributions; see [9] for some more numerical experiments. The minimal distance of 1 between clusters is easily explained by the fact that clusters separated by a distance less than 1 would be interacting and

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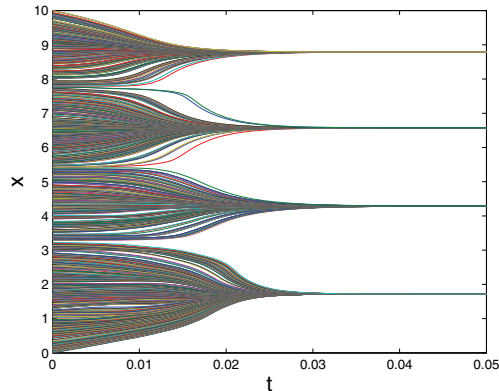


FIG. 1.1. Evolution with time of the values $x_i(t)$ for 1000 agents, with initial values randomly and uniformly distributed on $[0, 10]$. Observe the convergence to four clusters separated by a distance slightly greater than 2.

attracting each other. The observation that the typical intercluster distance is close to 2 is, however, more surprising. In this paper, we focus on understanding these convergence properties and the structure of the set of clusters, including the asymptotic behavior for large n .

Note that the agent interaction topology in (1.1) *explicitly depends on the agent states*, as $x_j(t)$ influences $x_i(t+1)$ only if $|x_i(t) - x_j(t)| < 1$. Many multiagent systems involve a changing interaction topology; see, e.g., [1, 10, 11, 16, 19, 20] and [17, 18] for surveys. In some cases, the interaction topology evolves randomly or according to some exogenous scheme, but in other cases it is modeled as a function of the agent states. The latter is typically the case for models of animals or robots with limited visibility. This state dependence, however, is usually not taken into account in the analysis, probably due to the technical difficulties that it presents. Notable exceptions include [6, 7, 12], in which the authors consider second order multiagent systems where the agent velocities or headings are influenced by those of all the other agents, and where the intensity of these influences varies with the distances between agents.

In an effort to understand the effect of state-dependent interaction topologies, we have recently analyzed [3] one of the simplest discrete-time multiagent systems of this kind, namely, Krause's model¹ of opinion dynamics [13]: n agents maintain real numbers ("opinions") $x_i(t)$, $i = 1, \dots, n$, and synchronously update them as follows:

$$x_i(t+1) = \frac{\sum_{j: |x_i(t) - x_j(t)| < 1} x_j(t)}{\sum_{j: |x_i(t) - x_j(t)| < 1} 1}.$$

This model was particularly appealing due to its simple formulation, and due to some peculiar behaviors that it exhibits, which cannot be explained without taking into account the explicit dynamics of the interaction topology. Indeed, a first analysis using results on infinite inhomogeneous matrix products, as in [10, 14], shows convergence to clusters in which all agents share the same opinion and shows that the distance between any two clusters is at least 1. Numerical simulations, however, show a qualitative behavior similar to the one shown in Figure 1.1 for the model (1.1): the

¹The model is sometimes referred to as the Hegselmann–Krause model.

distance between adjacent clusters is usually significantly larger than 1, and typically close to 2, when the number of agents is sufficiently large, a phenomenon for which no explanation was available.

Our goal in [3] was thus to develop a deeper understanding of Krause's model and of these observed phenomena by using explicitly the dynamics of the interaction topology. To this effect, we introduced a new notion of stability, tailored to such multiagent systems, which provided an explanation for the observed intercluster distances when the number of agents is large. Furthermore, to understand the asymptotic behavior as the number of agents increases, we also studied a model involving a continuum of agents. We obtained partial convergence results for this continuum model, and proved nontrivial lower bounds on the intercluster distances, under some conditions.

Our results in [3] were, however, incomplete in certain respects. In particular, the question of convergence of the continuum model remains open, and some of the results involve assumptions that are not easy to check a priori. We see two main reasons for these difficulties. First, the system is asymmetric, in the sense that the influence of $x_j(t)$ on $x_i(t+1)$ can be very different from that on $x_j(t+1)$, when i and j do not have the same number of neighbors. Second, the discrete-time nature of the system allows, for the continuum model, buildup of an infinite concentration of agents with the same opinion, thus breaking the continuity of the agent distribution.

For the above reasons, we have chosen to analyze here the system (1.1), a continuous-time symmetric variant of Krause's model, for which we provide crisper and more complete results. One reason is that, thanks to the symmetry, the average value $\frac{1}{n} \sum_i x_i(t)$ is preserved, and the average value of a group of agents evolves independently of the interactions taking place within the group, unlike Krause's model. In addition, when two agent values approach each other, their relative velocity decays to zero, preventing the formation of infinite concentration in finite time. The continuous-time nature of the system brings up, however, some new mathematical challenges, related, for example, to the existence and uniqueness of solutions.

To summarize, the objective of the present paper is twofold. First, we want to advance our understanding of multiagent systems with state-dependent interactions by analyzing in full detail one simple but nontrivial such system. Second, we want to explain the convergence of agents to clusters separated by approximately twice the interaction distance, a phenomenon that often arises in such opinion dynamics models (see [3, 9, 15] and the references therein).

1.1. Outline and contributions. In section 2, we give some basic properties of the model (1.1) and prove convergence to clusters in which all agents share the same value. We then analyze the distance between adjacent clusters, building on an appropriate notion of stability with respect to perturbing agents introduced in [3]. This analysis leads to a necessary and sufficient condition for stability that is consistent with the experimentally observed intercluster distances, and to a conjecture that the probability of convergence to a stable equilibrium tends to one as the number of agents increases. In section 3, we introduce a variant involving a continuum of agents to approximate the model for the case of a finite but large number of agents. Under some smoothness assumptions on the initial conditions, we prove the existence of a unique solution, convergence to clusters, and nontrivial lower bounds on the intercluster distances, consistent with the necessary and sufficient condition for stability in the discrete-agent model. Finally, in section 4, we explore the relation between the two models, and establish that the behavior of the discrete model approaches that of the continuum model over finite but arbitrarily long time intervals, provided that the

number of agents is sufficiently large.

The results summarized above differ from those obtained in [3] for Krause's model in three respects: (i) we prove the convergence of the continuum model, in contrast to the partial results obtained for Krause's model; (ii) all of our stability and approximation results are valid under some simple and easily checkable smoothness assumptions on the initial conditions, unlike the corresponding results in [3], which require, for example, the distance between the largest and smallest opinions to remain larger than 2 at all times; (iii) finally, we settle the problem of existence and uniqueness of a solution to our equations, a problem that did not arise for Krause's discrete-time model.

1.2. Related work. Our model (1.1) is closely related to that treated by Canuto, Fagnani, and Tilli [5], who consider a continuum of multidimensional opinions, while treating discrete agents as a special case. In the case of discrete agents with one-dimensional opinions, the evolution is described by

$$\dot{x}_i(t) = \sum_j \xi(x_i(t) - x_j(t)) (x_i(t) - x_j(t)),$$

where ξ is a continuous nonnegative radially symmetric and decaying function, taking positive values only for arguments with norm smaller than a certain constant R . They also consider a first order discrete-time² approximation of their model, described in the case of discrete agents with one-dimensional opinions by $x_i(t+\delta) = x_i(t) + \delta\dot{x}_i(t)$. Our model is therefore a particular case of their continuous-time model in one dimension, with a step function for ξ , except that a step function does not satisfy their continuity assumption.

The authors of [5] prove convergence of the opinions, in distribution, to clusters separated by at least R for both discrete- and continuous-time models, but do not address the issue of intercluster distances being significantly larger than R and close to $2R$. Their convergence proof relies on the decrease of the measured variance of the opinion distribution, and is based on an Eulerian representation that follows the density of agent opinions, in contrast to the Lagrangian representation used in this paper, which follows the opinion x of each agent. It is interesting to note that, despite differences between these two methods for proving convergence, they both appear to fail in the absence of symmetry and cannot be used to prove convergence for the continuum-agent variant of Krause's model.

Finally, the models in this paper are related to other classes of rendezvous methods and opinion dynamics models, as described in [3, 15] and the references therein. Several more complex decentralized control laws are built on such rendezvous methods.

2. Discrete agents. The differential equation (1.1) usually has no differentiable solutions. Indeed, observe that the right-hand side of the equation can be discontinuous in time when the interaction topology changes, which can prevent x from being differentiable. To avoid this difficulty, we consider functions $x : \mathfrak{R}^+ \rightarrow \mathfrak{R}^n$ that are solutions of the integral version of (1.1), namely,

$$(2.1) \quad x_i(t) = x_i(0) + \int_0^t \sum_{j: |x_i(\tau) - x_j(\tau)| < 1} (x_j(\tau) - x_i(\tau)) d\tau.$$

Note that, for all t at which $\dot{x}_i(t)$ exists, it can be computed using (1.1).

²The assumption on the continuity of ξ appears unnecessary in the discrete-time case, as was recently confirmed by one of the authors of [5] in a personal communication.

2.1. Existence and convergence. Time-switched linear systems are of the form $x(t) = x(0) + \int_0^t A_\tau x(\tau) d\tau$, where the matrix A_t is a piecewise constant function of t . They always admit a unique solution, provided that the number of switches taking place during any finite time interval is finite. Position-switched systems of the form $\dot{x}(t) = x(0) + \int_0^t A_{x(\tau)} x(\tau) d\tau$ may, on the other hand, admit no or multiple solutions. Our model (2.1) belongs to the latter class and indeed admits multiple solutions for some initial conditions. Observe, for example, that the two-agent system with initial condition $\tilde{x} = (-\frac{1}{2}, \frac{1}{2})$ admits a first solution $x(t) = \tilde{x}$ and a second solution $x(t) = \tilde{x}e^{-t}$. The latter solution indeed satisfies the differential equation (1.1) at every time except 0 and thus satisfies (2.1). We will see, however, that such cases are exceptional.

We say that $\tilde{x} \in \mathfrak{R}^n$ is a *proper initial condition* of (2.1) if the following hold:

(a) There exists a unique $x : \mathfrak{R}^+ \rightarrow \mathfrak{R}^n : t \rightarrow x(t)$ satisfying (2.1) and such that $x(0) = \tilde{x}$.

(b) The subset of \mathfrak{R}^+ on which x is not differentiable is at most countable and has no accumulation points.

(c) If $x_i(t) = x_j(t)$ holds for some t , then $x_i(t') = x_j(t')$ for every $t' \geq t$.

We then say that the solution x is a *proper solution* of (2.1). The proof of the following result is sketched in Appendix A, and a detailed version is available in [4].

THEOREM 1. *Almost all $\tilde{x} \in \mathfrak{R}^n$ (in the sense of Lebesgue measure) are proper initial conditions.*

It follows from condition (c) and from the continuity of proper solutions that if $x_i(t) \geq x_j(t)$ holds for some t , then this inequality holds for all subsequent times. For the sake of clarity, we thus assume in what follows that the components of proper initial conditions are sorted; that is, if $i > j$, then $\tilde{x}_i \geq \tilde{x}_j$, which also implies that $x_i(t) \geq x_j(t)$ for all t . Moreover, an explicit computation, which we perform in section 3 for a more complex system, shows that $|x_i(t) - x_j(t)| \geq |\tilde{x}_i - \tilde{x}_j| e^{-nt}$. Observe finally that if $x_{i+1}(t^*) - x_i(t^*) > 1$ holds for some t^* for a proper solution x , then $\dot{x}_{i+1}(t) \geq 0$ and $\dot{x}_i(t) \leq 0$ hold for almost all subsequent t , so that $x_{i+1}(t) - x_i(t)$ remains larger³ than 1. As a consequence, the system can then be decomposed into two independent subsystems, consisting of agents $1, \dots, i$, and $i + 1, \dots, n$, respectively.

We now characterize the evolution of the average and variance (sum of squared differences from the average) of the opinions. For this purpose, we let F be the set of vectors $\tilde{s} \in \mathfrak{R}^n$ such that, for all $i, j \in \{1, \dots, n\}$, either $\tilde{s}_i = \tilde{s}_j$ or $|\tilde{s}_i - \tilde{s}_j| \geq 1$. We refer to vectors in F as *equilibria*.

PROPOSITION 1. *Let x be a proper solution of (2.1). The average opinion $\bar{x}(t) = \frac{1}{n} \sum_{i=1}^n x_i(t)$ is constant. The sum of squared differences from the average, $V(x(t)) = \sum_{i=1}^n (x_i(t) - \bar{x}(t))^2$, is nonincreasing. Furthermore, with the exception of a countable set of times, if $x_t \notin F$ (respectively, $x_t \in F$), then the derivative $(dV/dt)(x(t))$ is negative (respectively, zero).*

Proof. For all t , except possibly for countably many,

$$(2.2) \quad \frac{d}{dt} \bar{x}(t) = \frac{1}{n} \sum_i \dot{x}_i(t) = \frac{1}{n} \sum_{(i,j): |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t)) = 0.$$

Since $x(t)$ is continuous, this implies that $\bar{x}(t)$ is constant.

³The case $x_{i+1}(t) - x_i(t) = 1$ is more complex. Agents could indeed become “reconnected,” as in the nonuniqueness example given above, because (2.1) is allowed to fail at countably many times.

Observe now that, for all t at which x is differentiable, $\frac{d}{dt}V(x(t))$ equals

$$\sum_{i=1}^n 2(x_i(t) - \bar{x}(t)) \dot{x}_i(t) = 2 \sum_{i=1}^n x_i(t) \dot{x}_i(t) = 2 \sum_{i=1}^n \sum_{j: |x_i(t) - x_j(t)| < 1} x_i(t) (x_j(t) - x_i(t)),$$

where we have used the relation (2.2) twice and the definition (2.1). The right-hand side of this equality can be rewritten as

$$\sum_{i,j: |x_i(t) - x_j(t)| < 1} x_i(t) (x_j(t) - x_i(t)) + \sum_{j,i: |x_j(t) - x_i(t)| < 1} x_j(t) (x_i(t) - x_j(t)),$$

so that

$$\frac{d}{dt}V(x(t)) = - \sum_{i,j: |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t))^2.$$

The latter expression is negative if $x(t) \notin F$ and zero otherwise. \square

There are several convergence proofs for the system (2.1). We present here a simple one, which highlights the importance of the average preservation and symmetry properties and extends nicely to the continuum model. A proof relying on other properties and that can be used in the absence of symmetry can be found in [9].

THEOREM 2. *Every proper solution x of (2.1) converges to a limit $x^* \in F$; that is, for any i, j , if $x_i^* \neq x_j^*$, then $|x_i^* - x_j^*| \geq 1$.*

Proof. Observe that, by symmetry, the equality

$$\sum_{i=1}^k \sum_{j \leq k: |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t)) = 0$$

holds for any k and any t . Therefore, it follows from (2.1) that for all t , except possibly for countably many,

$$(2.3) \quad \frac{d}{dt} \sum_{i=1}^k x_i(t) = \sum_{i=1}^k \sum_{j > k: |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t)),$$

which is nonnegative, because $j > k > i$ implies $x_j(t) - x_i(t) \geq 0$. Since $x_i(t) \leq \max_j x_j(0) = x_n(0)$ for all i and $t \geq 0$, $\sum_{i=1}^k x_i(t)$ is bounded and therefore converges monotonically for any k . It then follows that every $x_i(t)$ converges to a limit x_i^* . We assume that $x_k^* \neq x_{k+1}^*$ and suppose, to obtain a contradiction, that $x_{k+1}^* - x_k^* < 1$. Then, since every term $x_j(t) - x_i(t)$ on the right-hand side of (2.3) is nonnegative, the derivative on the left-hand side is asymptotically positive and bounded away from 0, preventing the convergence of $\sum_{i=1}^k x_i(t)$. Therefore, $x_{k+1}^* - x_k^* \geq 1$. \square

2.2. Stable equilibria and intercluster distances. By the term *clusters*, we will mean the limiting values to which the agent opinions converge. With some abuse of terminology, we also refer to a set of agents whose opinions converge to the same value as a cluster. Theorem 2 implies that clusters are separated by a distance of at least 1. On the other hand, extensive numerical experiments indicate that the distance between adjacent clusters is typically significantly greater than 1, and if the clusters contain the same number of agents, it is usually close to 2. We believe that this phenomenon can, at least partially, be explained by the fact that clusters that are too

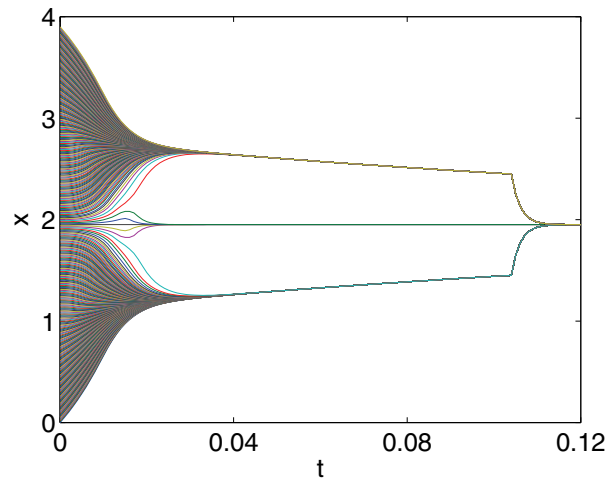


FIG. 2.1. Example of a temporary, “metastable” equilibrium. Initially, two clusters are formed and do not interact with each other, but they both interact with a small number of agents in between. As a result, the distance separating them eventually becomes less than 1. The clusters then attract each other directly and merge into a single, larger cluster.

close to each other can be forced to merge by the presence of a small number of agents between them, as in Figure 2.1. To formalize this idea we introduce a generalization of the system (2.1) in which each agent i has a weight w_i , and its opinion evolves according to

$$(2.4) \quad x_i(t) = x_i(0) + \int_0^t \sum_{j: |x_i(\tau) - x_j(\tau)| < 1} w_j (x_j(\tau) - x_i(\tau)) d\tau.$$

The results of section 2.1 carry over to the weighted case (the proof is the same). We will refer to the sum of the weights of all agents in a cluster as its *weight*. If all the agents in a cluster have exactly the same opinion, the cluster behaves as a single agent with this particular weight.⁴

Let $\tilde{s} \in F$ be an equilibrium vector. Suppose that we add a new agent of weight δ and initial opinion x_0 , consider the resulting configuration as an initial condition, and let the system evolve according to some solution $x(t)$ (we do not require uniqueness). We define $\Delta(\delta, \tilde{s})$ as the supremum of $|x_i(t) - \tilde{s}_i|$, where the supremum is taken over all possible initial opinions x_0 of the perturbing agent, all i , all times t , and all possible solutions $x(t)$ of the system (2.1). We say that \tilde{s} is *stable* if $\lim_{\delta \downarrow 0} \Delta(\delta, \tilde{s}) = 0$. An equilibrium is thus unstable if some modification of fixed size can be achieved by adding an agent of arbitrarily small weight. This notion of stability is almost the same as the one that we introduced for Krause’s model in [2, 3].

THEOREM 3. *An equilibrium is stable if and only if for any two clusters A and B with weights W_A and W_B , respectively, their distance is greater than $d = 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}$.*

⁴In the case of nonproper initial conditions leading to multiple solutions, it is not hard to show that there exists at least one solution in which each cluster behaves as a single agent with the corresponding weight.

Proof. The proof is very similar to the proof of Theorem 2 in [3]. The main idea is the following. A perturbing agent can initially be connected to at most two clusters, and cannot perturb the equilibrium substantially if it is connected to none or one. If it is connected to two clusters A, B , it moves in the direction of their center of mass $\frac{W_A \bar{s}_A + W_B \bar{s}_B}{W_A + W_B}$, while the two clusters move at a much slower pace, proportional to the perturbing agent's weight. We note that, by a simple algebraic calculation, the center of mass of two clusters is within unit distance from both clusters if and only if their distance is no greater than d .

If the distance between the two clusters is greater than d , then the center of mass of the two clusters is greater than unit distance away from one of the clusters, say, from B . Therefore, eventually the perturbing agent is no longer connected to B , and rapidly joins cluster A , having modified the cluster positions only proportionally to its weight. Thus, the equilibrium is stable.

On the other hand, if the distance between the two clusters is less than d , then the center of mass is less than unit distance away from both clusters. We can place the perturbing agent at the center of mass. Then, the perturbing agent does not move, but keeps attracting the two clusters, until eventually they become connected and then rapidly merge. Thus, the equilibrium is not stable.

If the distance between clusters is exactly equal to d , the center of mass is at exactly unit distance from one of the two clusters. Placing a perturbing agent at the center of mass results in nonunique solutions. In one of these solutions, the clusters start moving towards their center of mass, and the subsequent behavior is the same as in the case where the distance between clusters is less than d , thus again showing instability. Such a solution violates the differential version of (2.1) only at time $t = 0$ and thus satisfies (2.1). \square

Theorem 3 characterizes stable equilibria in terms of a lower bound on intercluster distances. It allows for intercluster distances at a stable equilibrium that are less than 2, provided that the clusters have different weights. This is consistent with experimental observations for certain initial opinion distributions (see [9], for example). On the other hand, for the frequently observed case of clusters with equal weights, stability requires intercluster distances of at least 2. Thus, this result comes close to a full explanation of the observed intercluster distances of about 2.2. Of course, there is no guarantee that our system will converge to a stable equilibrium. (A trivial example is obtained by initializing the system at an unstable equilibrium.) However, we have observed that for a given distribution of initial opinions, and as the number of agents increases, we almost always obtain convergence to a stable equilibrium. This leads us to the following conjecture.

CONJECTURE 1. *Suppose that the initial opinions are chosen randomly and independently according to a bounded probability density function with connected support, which is also bounded below by a positive number on its support. Then, the probability of convergence to a stable equilibrium tends to 1, as the number of agents increases to infinity.*

In addition to extensive numerical evidence (see, for example, [9]), this conjecture is supported by the intuitive idea that if the number of agents is sufficiently large, convergence to an unstable equilibrium is made impossible by the presence of at least one agent connected to the two clusters. It is also supported by results obtained in the next sections. A similar conjecture has been made for Krause's model [2, 3].

3. Agent continuum. To further analyze the properties of (2.1) and its behavior as the number of agents increases, we now consider a variant involving a continuum

of agents. We use the interval $I = [0, 1]$ to index the agents, and denote by Y the set of bounded measurable functions $\tilde{x} : I \rightarrow \mathfrak{R}$, attributing an opinion $\tilde{x}(\alpha) \in \mathfrak{R}$ to every agent in I . As an example, a uniform distribution of opinions is given by $\tilde{x}(\alpha) = \alpha$. We use the function $x : I \times \mathfrak{R}^+ \rightarrow \mathfrak{R} : (\alpha, t) \rightarrow x_t(\alpha)$ to describe the collection of all opinions at different times.⁵ We denote by x_t the function in Y obtained by restricting x to a certain value of t . For a given initial opinion function $\tilde{x}_0 \in Y$, we are interested in functions x satisfying

$$(3.1) \quad \frac{d}{dt}x_t(\alpha) = \int_{\beta: (\alpha, \beta) \in C_{x_t}} (x_t(\beta) - x_t(\alpha)) d\beta,$$

where $C_{\tilde{x}} \subseteq I^2$ is defined for any $\tilde{x} \in Y$ by

$$C_{\tilde{x}} := \{(\alpha, \beta) \in I^2 : |\tilde{x}(\alpha) - \tilde{x}(\beta)| < 1\}.$$

In what follows, we denote by $\chi_{\tilde{x}}$ the indicator function of $C_{\tilde{x}}$.

Note that x_0 , the restriction of x to $t = 0$, should not be confused with \tilde{x}_0 , an arbitrary function in Y intended as an initial condition but for which there may possibly exist no or several corresponding functions x . The existence or uniqueness of a solution to (3.1) is not guaranteed, and there may, moreover, exist functions that satisfy this equation in a weaker sense, without being differentiable in t . For this reason, it is more convenient to formally define the model through an integral equation. For an initial opinion function $\tilde{x}_0 \in Y$, we are interested in measurable functions $x : I \times \mathfrak{R}^+ \rightarrow \mathfrak{R} : (\alpha, t) \rightarrow x_t(\alpha)$ such that

$$(3.2) \quad x_t(\alpha) = \tilde{x}_0(\alpha) + \int_0^t \left(\int_{\beta: (\alpha, \beta) \in C_{x_\tau}} (x_\tau(\beta) - x_\tau(\alpha)) d\beta \right) d\tau$$

holds for every t and for every $\alpha \in I$.⁶ Similar to the case of discrete agents, one can easily prove that, for any solution x of (3.2), $\bar{x}_t := \int_0^1 x_t(\alpha) d\alpha$ is constant, and $\int_0^1 (x_t(\alpha) - \bar{x}_t)^2 d\alpha$ is nonincreasing in t .

For the sake of simplicity, we will restrict our attention to nondecreasing (and often increasing) opinion functions and define X as the set of nondecreasing bounded functions $\tilde{x} : I \rightarrow \mathfrak{R}$. This is no essential loss of generality, because the only quantities of interest relate to the distribution of opinions; furthermore, monotonicity of initial opinion functions can be enforced using a measure-preserving reindexing of the agents; finally, monotonicity is preserved by the dynamics under mild conditions. In what follows, an element of X will be referred to as a nondecreasing function. Furthermore, if $x : I \times [0, \infty) \rightarrow \mathfrak{R}$ is such that $x_t \in X$ for all t , we will also say that x is nondecreasing.

3.1. Existence and uniqueness of solutions. The existence of a unique solution to (3.2) is in general not guaranteed, as there exist initial conditions allowing for multiple solutions. Consider, for example, $\tilde{x}_0(\alpha) = -1/2$ if $\alpha \in [0, \frac{1}{2}]$, and $\tilde{x}_0(\alpha) = 1/2$ otherwise. Observe that, similar to our discrete-agent example, $x_t = \tilde{x}_0$ and $x_t(\alpha) = \tilde{x}_0(\alpha)e^{-t/2}$ are two possible solutions of (3.2). Nevertheless, we will prove

⁵Note the reversal of notational conventions: the subscript now indicates time rather than an agent's index.

⁶A slightly more general definition would require (3.2) to be satisfied for almost all $\alpha \in I$. However, this would result in distracting technicalities.

existence and uniqueness, provided that the initial condition, as a function of α , has a positive and bounded increase rate; this is equivalent to assuming that the density of initial opinions is bounded from above and from below on its support, which is connected.

Our proof of existence and uniqueness is based on the Banach fixed point theorem, applied to the operator G that maps measurable functions $x : I \times [0, t_1] \rightarrow \mathfrak{R}$ into the set of such functions, according to

$$(G(x))_t(\alpha) = \tilde{x}_0(\alpha) + \int_0^t \left(\int_{\beta: (\alpha, \beta) \in C_{x_\tau}} (x_\tau(\beta) - x_\tau(\alpha)) \, d\beta \right) d\tau$$

for some fixed t_1 . Observe indeed that x is a solution of the system (3.2) if and only if $x_0 = \tilde{x}$ and $x = G(x)$.

It is convenient to introduce some additional notation. For positive real numbers m, M , we call $X_m \subset X$ the set of nondecreasing functions $\tilde{x} : I \rightarrow \mathfrak{R}$ such that

$$\frac{\tilde{x}(\beta) - \tilde{x}(\alpha)}{\beta - \alpha} \geq m$$

holds for every $\beta \neq \alpha$, and we call $X^M \subset X$ the set of nondecreasing functions \tilde{x} such that

$$\frac{\tilde{x}(\beta) - \tilde{x}(\alpha)}{\beta - \alpha} \leq M$$

holds for every $\beta \neq \alpha$. We then denote $X_m \cap X^M$ by X_m^M and say that a function $\tilde{x} \in X$ is *regular* if it belongs to X_m^M for some $m, M > 0$. Now let \mathcal{L} be the operator defined on X and taking its values in the set of functions from I to \mathfrak{R} , defined by

$$(3.3) \quad \mathcal{L}(\tilde{x})(\alpha) = \int \chi_{\tilde{x}}(\alpha, \gamma) (\tilde{x}(\gamma) - \tilde{x}(\alpha)) \, d\gamma.$$

Observe that (3.2) can be rewritten as $x_t(\alpha) = \tilde{x}_0(\alpha) + \int_0^t \mathcal{L}(x_\tau)(\alpha) \, d\tau = (G(x))_t$.

The proof of existence and uniqueness rests on two important qualitative properties of our model. The first, given in Lemma 1, establishes that \mathcal{L} is Lipschitz continuous on X_m . This property will allow us to establish that the operator G is a contraction (when t_1 is small enough) and to apply Banach’s fixed point theorem. The second, Lemma 2, gives bounds on the rate at which the opinions of different agents can approach each other. It is instrumental in showing that regularity is preserved, allowing us to apply the same argument and extend the solution to arbitrarily long time intervals.

LEMMA 1. *Let \tilde{x} be a function in X_m , where $m > 0$. The operator \mathcal{L} is Lipschitz continuous at \tilde{x} with respect to the $\|\cdot\|_\infty$ norm. More precisely, for any $\tilde{y} \in Y$,*

$$\|\mathcal{L}(\tilde{x}) - \mathcal{L}(\tilde{y})\|_\infty \leq \left(2 + \frac{8}{m} \right) \|\tilde{x} - \tilde{y}\|_\infty.$$

Proof. Let $\tilde{x} \in X_m$, $\tilde{y} \in Y$, and $\delta = \|\tilde{x} - \tilde{y}\|_\infty$. Fix some $\alpha \in I$, and let $N_x := \{\gamma : |\tilde{x}(\gamma) - \tilde{x}(\alpha)| < 1\}$, $N_y := \{\gamma : |\tilde{y}(\gamma) - \tilde{y}(\alpha)| < 1\}$ be the sets of agents connected to α under the configuration defined by \tilde{x} and \tilde{y} , respectively. Let also $N_{xy} = N_x \cap N_y$, $N_{x \setminus y} = N_x \setminus N_{xy}$, and $N_{y \setminus x} = N_y \setminus N_{xy}$. By the definition (3.3)

of \mathcal{L} , we have $\mathcal{L}(\tilde{x})(\alpha) = \int_{N_x} (\tilde{x}(\gamma) - \tilde{x}(\alpha)) d\gamma$ and $\mathcal{L}(\tilde{y})(\alpha) = \int_{N_y} (\tilde{y}(\gamma) - \tilde{y}(\alpha)) d\gamma$. Therefore,

$$\begin{aligned} \mathcal{L}(\tilde{y})(\alpha) - \mathcal{L}(\tilde{x})(\alpha) &= \int_{N_{xy}} (\tilde{y}(\gamma) - \tilde{x}(\gamma) - \tilde{y}(\alpha) + \tilde{x}(\alpha)) d\gamma \\ &\quad + \int_{N_{y \setminus x}} (\tilde{y}(\gamma) - \tilde{y}(\alpha)) d\gamma - \int_{N_{x \setminus y}} (\tilde{x}(\gamma) - \tilde{x}(\alpha)) d\gamma. \end{aligned}$$

It follows from the definitions of N_x and N_y that $|\tilde{x}(\gamma) - \tilde{x}(\alpha)| < 1$ holds for every $\gamma \in N_{x \setminus y} \subseteq N_x$, and $|\tilde{y}(\gamma) - \tilde{y}(\alpha)| < 1$ holds for every $\gamma \in N_{y \setminus x} \subseteq N_y$. This leads to

(3.4)

$$\begin{aligned} |\mathcal{L}(\tilde{y})(\alpha) - \mathcal{L}(\tilde{x})(\alpha)| &\leq \int_{N_{xy}} (|\tilde{y}(\gamma) - \tilde{x}(\gamma)| + |\tilde{y}(\alpha) - \tilde{x}(\alpha)|) d\gamma + |N_{x \setminus y}| + |N_{y \setminus x}| \\ &\leq 2|N_{xy}| \delta + |N_{x \setminus y}| + |N_{y \setminus x}| \\ &\leq 2\delta + |N_{x \setminus y}| + |N_{y \setminus x}|, \end{aligned}$$

where we have used the bound $|N_{xy}| \leq |I| = 1$ to obtain the last inequality. It remains to give bounds on $|N_{x \setminus y}|$ and $|N_{y \setminus x}|$.

If $\gamma \in N_{y \setminus x}$, then $\gamma \in N_y$, and $|\tilde{y}(\gamma) - \tilde{y}(\alpha)| < 1$. This implies that

$$|\tilde{x}(\gamma) - \tilde{x}(\alpha)| \leq |\tilde{x}(\gamma) - \tilde{y}(\gamma)| + |\tilde{y}(\gamma) - \tilde{y}(\alpha)| + |\tilde{y}(\alpha) - \tilde{x}(\alpha)| \leq \delta + 1 + \delta.$$

Since such a γ does not belong to N_x , we also have $|\tilde{x}(\gamma) - \tilde{x}(\alpha)| \geq 1$. Thus, for every $\gamma \in N_{y \setminus x}$, the opinion $\tilde{x}(\gamma)$ lies in the set

$$[\tilde{x}(\alpha) - 1 - 2\delta, \tilde{x}(\alpha) - 1] \cup [\tilde{x}(\alpha) + 1, \tilde{x}(\alpha) + 1 + 2\delta],$$

which has length at most 4δ . Since the rate of change of opinions (with respect to the index γ) is at least m , we conclude that $|N_{y \setminus x}| \leq 4\delta/m$. A similar argument shows that $|N_{x \setminus y}| \leq 4\delta/m$. The inequality (3.4) then becomes

$$|\mathcal{L}(\tilde{y})(\alpha) - \mathcal{L}(\tilde{x})(\alpha)| \leq 2\delta + 8\frac{\delta}{m} = \left(2 + \frac{8}{m}\right) \|\tilde{y} - \tilde{x}\|_\infty,$$

which is the desired result. \square

LEMMA 2. Let $\tilde{x} \in Y$. Suppose that $\alpha, \beta \in I$, and $x(\alpha) \leq x(\beta)$. Then,

$$\mathcal{L}(\tilde{x})(\beta) - \mathcal{L}(\tilde{x})(\alpha) \geq -(\tilde{x}(\beta) - \tilde{x}(\alpha)).$$

Furthermore, if $\tilde{x} \in X_m$ for some $m > 0$, then

$$\mathcal{L}(\tilde{x})(\beta) - \mathcal{L}(\tilde{x})(\alpha) \leq \frac{2}{m} (\tilde{x}(\beta) - \tilde{x}(\alpha)).$$

Proof. Let $N_\alpha := \{\gamma : |\tilde{x}(\gamma) - \tilde{x}(\alpha)| < 1\}$ and $N_\beta := \{\gamma : |\tilde{x}(\gamma) - \tilde{x}(\beta)| < 1\}$ be the sets of agents connected to α and β , respectively. Now let $N_{\alpha\beta} = N_\alpha \cap N_\beta$, $N_{\alpha \setminus \beta} = N_\alpha \setminus N_{\alpha\beta}$, and $N_{\beta \setminus \alpha} = N_\beta \setminus N_{\alpha\beta}$. It follows from the definition (3.3) of \mathcal{L} that

$$\begin{aligned} \mathcal{L}(\tilde{x})(\beta) &= \int_{N_{\alpha\beta}} (\tilde{x}(\gamma) - \tilde{x}(\beta)) d\gamma + \int_{N_{\beta \setminus \alpha}} (\tilde{x}(\gamma) - \tilde{x}(\beta)) d\gamma, \\ \mathcal{L}(\tilde{x})(\alpha) &= \int_{N_{\alpha\beta}} (\tilde{x}(\gamma) - \tilde{x}(\alpha)) d\gamma + \int_{N_{\alpha \setminus \beta}} (\tilde{x}(\gamma) - \tilde{x}(\alpha)) d\gamma. \end{aligned} \tag{3.5}$$

The definitions of the sets $N_{\beta \setminus \alpha}$ and $N_{\alpha \setminus \beta}$, together with $\tilde{x}(\beta) \geq \tilde{x}(\alpha)$, imply that $\tilde{x}(\gamma) > \tilde{x}(\alpha)$ holds for all $\gamma \in N_{\beta \setminus \alpha}$, and $\tilde{x}(\gamma) < \tilde{x}(\beta)$ holds for every $\gamma \in N_{\alpha \setminus \beta}$. Using these inequalities and subtracting the two equalities above, we obtain

$$\mathcal{L}(\tilde{x})(\beta) - \mathcal{L}(\tilde{x})(\alpha) \geq \int_{N_{\alpha\beta}} (\tilde{x}(\alpha) - \tilde{x}(\beta)) d\gamma + \int_{N_{\beta \setminus \alpha} \cup N_{\alpha \setminus \beta}} (\tilde{x}(\alpha) - \tilde{x}(\beta)) d\gamma.$$

Since $|N_{\alpha\beta}| + |N_{\alpha \setminus \beta}| + |N_{\beta \setminus \alpha}| = |N_{\alpha} \cup N_{\beta}| \leq |I| = 1$, we obtain the first part of the lemma.

Let us now assume that $\tilde{x} \in X_m$. It follows from (3.5) and from the inequality $\tilde{x}(\beta) \geq \tilde{x}(\alpha)$ that

$$(3.6) \quad \mathcal{L}(\tilde{x})(\beta) - \mathcal{L}(\tilde{x})(\alpha) \leq \int_{N_{\beta \setminus \alpha}} (\tilde{x}(\gamma) - \tilde{x}(\beta)) d\gamma - \int_{N_{\alpha \setminus \beta}} (\tilde{x}(\gamma) - \tilde{x}(\alpha)) d\gamma,$$

which is bounded by $|N_{\beta \setminus \alpha}| + |N_{\alpha \setminus \beta}|$. Observe that $\tilde{x}(N_{\beta \setminus \alpha}) \subseteq [\tilde{x}(\alpha) + 1, \tilde{x}(\beta) + 1]$. Since $\tilde{x} \in X_m$, we have

$$|N_{\beta \setminus \alpha}| \leq \frac{1}{m} |\tilde{x}(N_{\beta \setminus \alpha})| = \frac{1}{m} (\tilde{x}(\beta) - \tilde{x}(\alpha)).$$

The same bound holds on $|N_{\alpha \setminus \beta}|$. The second part of the lemma then follows from the bound (3.6). \square

The first part of Lemma 2 implies that, for any solution x , the difference of the opinions of two agents decreases at most exponentially fast. As a consequence, if the initial condition of x is an increasing function of α , then x_t is also increasing for all t . We note that this last property does not necessarily hold for nonregular initial conditions that are only nondecreasing.

We can now formally state our existence and uniqueness result, with the rest of the proof given in Appendix B. This result also shows that if the initial condition is regular, then the two models given by a differential or integral equation, respectively, admit a unique and common solution, which is regular at all times.

THEOREM 4. *Suppose that the initial opinion function satisfies $\tilde{x}_0 \in X_m^M$ for some $m, M > 0$. Then the models (3.1) and (3.2) admit a unique and common solution x , and x satisfies*

$$(3.7) \quad me^{-t} \leq \frac{x_t(\beta) - x_t(\alpha)}{\beta - \alpha} \leq Me^{4t/m}$$

for every t and $\beta \neq \alpha$.

3.2. Convergence and fixed points. In this section, we prove that opinions converge to clusters separated by at least unit distance, as in the case of discrete agents. The proof has some similarities with the one of Theorem 2. It involves three partial results, the first of which establishes the convergence of the average value of x_t on any interval. Lemma 3 involves an assumption that x_t is nondecreasing. By Theorem 4, this is guaranteed if the initial condition is regular.

LEMMA 3. *Let x be a nondecreasing solution of the integral equation (3.2). For any $c \in I$, the limit*

$$\lim_{t \rightarrow \infty} \int_0^c x_t(\alpha) d\alpha$$

exists. As a result, the average value $(\int_b^c x_t(\alpha) d\alpha)/(c-b)$ of x_t on any positive length interval $[b, c]$ converges as $t \rightarrow \infty$.

Proof. Fix some $c \in [0, 1]$ and t_1, t_2 with $0 \leq t_1 < t_2$. The evolution equation (3.2) yields

$$(3.8) \quad \int_0^c x_{t_2}(\alpha) d\alpha = \int_0^c x_{t_1}(\alpha) d\alpha + \int_{t_1}^{t_2} \left(\int_0^c \int_0^1 \chi_{x_\tau}(\alpha, \beta) (x_\tau(\beta) - x_\tau(\alpha)) d\beta d\alpha \right) d\tau,$$

where we have used the Fubini theorem to interchange the integration with respect to τ and α . We observe that

$$\int_0^c \int_0^c \chi_{x_\tau}(\alpha, \beta) (x_\tau(\beta) - x_\tau(\alpha)) d\beta d\alpha = 0,$$

because of the symmetry property $\chi_{x_\tau}(\alpha, \beta) = \chi_{x_\tau}(\beta, \alpha)$. Therefore,

$$(3.9) \quad \int_0^c \int_0^1 \chi_{x_\tau}(\alpha, \beta) (x_\tau(\beta) - x_\tau(\alpha)) d\beta d\alpha = \int_0^c \int_c^1 \chi_{x_\tau}(\alpha, \beta) (x_\tau(\beta) - x_\tau(\alpha)) d\beta d\alpha.$$

The latter integral is nonnegative, because $x_\tau(\beta) - x_\tau(\alpha) \geq 0$ whenever $\alpha \leq c \leq \beta$. Thus, $\int_0^c x_t(\alpha) d\alpha$ is a bounded and nondecreasing function of t , and hence converges, which is the desired result. \square

PROPOSITION 2. *Let x be a solution of the integral equation (3.2) such that x_t is nondecreasing in α for all t . For all $\alpha \in I$, except possibly for a countable set, the limit $\lim_{t \rightarrow \infty} x_t(\alpha)$ exists.*

Proof. Let $\tilde{y}(\alpha) = \limsup_{t \rightarrow \infty} x_t(\alpha)$. Since, for any t , $x_t(\alpha)$ is a nondecreasing function of α , it follows that $\tilde{y}(\alpha)$ is also nondecreasing in α . Let S be the set of all α at which $\tilde{y}(\cdot)$ is discontinuous. Since $\tilde{y}(\cdot)$ is nondecreasing, it follows that S is at most countable.

Fix some $\alpha \notin S$. Suppose, in order to derive a contradiction, that $x_t(\alpha)$ does not converge to $\tilde{y}(\alpha)$. We can then fix some $\epsilon > 0$ and a sequence of times t_n that converges to infinity, such that $x_{t_n}(\alpha) \leq \tilde{y}(\alpha) - \epsilon$. In particular, for any $\delta > 0$, we have

$$(3.10) \quad \int_{\alpha-\delta}^\alpha x_{t_n}(\beta) d\beta \leq \int_{\alpha-\delta}^\alpha x_{t_n}(\alpha) d\beta = \delta x_{t_n}(\alpha) \leq \delta \tilde{y}(\alpha) - \delta \epsilon.$$

Since $\tilde{y}(\cdot)$ is continuous at α , we can choose δ so that $\tilde{y}(\alpha - \delta) \geq \tilde{y}(\alpha) - \epsilon/3$. Furthermore, there exists a sequence of times τ_n that converges to infinity and such that

$$x_{\tau_n}(\alpha - \delta) \geq \tilde{y}(\alpha - \delta) - \frac{\epsilon}{3} \geq \tilde{y}(\alpha) - \frac{2\epsilon}{3}.$$

At those times, we have

$$(3.11) \quad \int_{\alpha-\delta}^\alpha x_{\tau_n}(\beta) d\beta \geq \int_{\alpha-\delta}^\alpha x_{\tau_n}(\alpha - \delta) d\beta = \delta x_{\tau_n}(\alpha - \delta) \geq \delta \tilde{y}(\alpha) - \frac{2\delta\epsilon}{3}.$$

However, (3.10) and (3.11) contradict the fact that $\int_{\alpha-\delta}^\alpha x_t(\alpha) d\alpha$ converges, thus establishing the desired result. \square

We now characterize the fixed points and the possible limit points of the system. Let $F \subset X$ be the set of nondecreasing functions \tilde{s} such that, for every $\alpha, \beta \in I$, either $\tilde{s}(\alpha) = \tilde{s}(\beta)$ or $|\tilde{s}(\alpha) - \tilde{s}(\beta)| > 1$. Similarly, let \overline{F} be the set of nondecreasing functions \tilde{s} such that, for almost every pair $(\alpha, \beta) \in I^2$, either $\tilde{s}(\alpha) = \tilde{s}(\beta)$ or $|\tilde{s}(\alpha) - \tilde{s}(\beta)| \geq 1$. Finally, we say that $\tilde{s} \in X$ is a *fixed point* if the integral equation (3.2) with initial condition \tilde{s} admits a unique solution $x_t = \tilde{s}$ for all t .

PROPOSITION 3.

- (a) Let x be a nondecreasing (in α , for all t) solution of the integral equation (3.2), and suppose that $\tilde{y}(\alpha) = \lim_{t \rightarrow \infty} x_t(\alpha)$ a.e. Then, $\tilde{y} \in \overline{F}$.
- (b) If $\tilde{s} \in F$, then \tilde{s} is a fixed point.
- (c) If \tilde{s} is a nondecreasing fixed point, then $\tilde{s} \in \overline{F}$.

Proof. (a) We take the limit in (3.8) as $t_2 \rightarrow \infty$. Since the left-hand side converges and the integral inside the brackets is nonnegative (by (3.9)), it follows that

$$\liminf_{\tau \rightarrow \infty} \int_0^c \int_0^1 \chi_{x_\tau}(\alpha, \beta) (x_\tau(\beta) - x_\tau(\alpha)) d\beta d\alpha = 0.$$

Using (3.9) and then Fatou's lemma, we obtain

$$\int_0^c \int_c^1 \liminf_{\tau \rightarrow \infty} \chi_{x_\tau}(\alpha, \beta) (x_\tau(\beta) - x_\tau(\alpha)) d\beta d\alpha = 0.$$

Note that $x_\tau(\beta) - x_\tau(\alpha)$ converges to $\tilde{y}(\beta) - \tilde{y}(\alpha)$ a.e. If $\chi_{\tilde{y}}(\alpha, \beta) = 1$, then $\chi_{x_\tau}(\alpha, \beta) = 1$ for τ large enough. This shows that $\liminf_{\tau \rightarrow \infty} \chi_{x_\tau}(\alpha, \beta) \geq \chi_{\tilde{y}}(\alpha, \beta)$. We conclude that

$$\int_0^c \int_c^1 \chi_{\tilde{y}}(\alpha, \beta) (\tilde{y}(\beta) - \tilde{y}(\alpha)) d\beta d\alpha = 0.$$

We integrate this equation over all $c \in [0, 1]$, interchange the order of integration, and obtain

$$\int_0^1 \int_\alpha^1 \chi_{\tilde{y}}(\alpha, \beta) (\tilde{y}(\beta) - \tilde{y}(\alpha)) (\beta - \alpha) d\beta d\alpha = 0.$$

This implies that, for almost all pairs (α, β) with $\alpha < \beta$ (with respect to the two-dimensional Lebesgue measure), we have $\chi_{\tilde{y}}(\alpha, \beta) (\tilde{y}(\beta) - \tilde{y}(\alpha)) = 0$, and either $\tilde{y}(\alpha) = \tilde{y}(\beta)$ or $\tilde{y}(\beta) \geq \tilde{y}(\alpha) + 1$. This is possible only if $\tilde{y} \in \overline{F}$ (the details of this last step are elementary and are omitted).

(b) Suppose that $\tilde{s} \in F$. We have either $\chi_{\tilde{s}}(\alpha, \beta) = 0$ or $\tilde{s}(\alpha) = \tilde{s}(\beta)$. Thus, $\int \chi_{\tilde{s}}(\alpha, \beta) (\tilde{s}(\beta) - \tilde{s}(\alpha)) d\beta = 0$ for all α , and $x_t = \tilde{s}$ for all t is thus a solution of the system. We now prove that this solution is unique. (Recall that uniqueness is part of our definition of a fixed point.)

Since \tilde{s} is bounded and belongs to F , there exists a positive $\epsilon < 1/2$ such that, for all $\alpha, \beta \in I$, either $\tilde{s}(\alpha) = \tilde{s}(\beta)$ or $|\tilde{s}(\alpha) - \tilde{s}(\beta)| > 1 + 3\epsilon$. Now let y be a solution of (3.2) with \tilde{s} as initial condition. Equation (3.2) readily implies that $|y_t(\alpha) - \tilde{s}(\alpha)| \leq \epsilon$ for all $t \in [0, \epsilon]$ and $\alpha \in I$. Therefore, for $t \in [0, \epsilon]$, there holds that $|y_t(\alpha) - y_t(\beta)| < 1$ if and only if $|\tilde{s}(\alpha) - \tilde{s}(\beta)| < 1$, and y_t is also a solution of the integral equation

$$y_t(\alpha) = \tilde{s}(\alpha) + \int_{\tau=0}^t \left(\int_{\beta: (\alpha, \beta) \in C_{\tilde{s}}} (y_\tau(\beta) - y_\tau(\alpha)) d\tau \right),$$

which unlike (3.2) is a linear system, because $C_{\tilde{s}}$ is constant. It can be shown, using, for example, the Lipschitz continuity of the corresponding linear operator, that this system admits a unique solution, so that $y_t = \tilde{s}$ holds for $t \in [0, \epsilon]$. Repeating this reasoning, we obtain $y_t = \tilde{s}$ for all $t > 0$, and \tilde{s} is thus a fixed point.

(c) Suppose that \tilde{s} is a nondecreasing fixed point. By the definition of a fixed point, the function x defined by $x_t = \tilde{s}$ for all t is a solution of the integral equation (3.2). Since it trivially converges to \tilde{s} and remains nondecreasing, the result follows from part (a) of this proposition. \square

The following theorem combines and summarizes the convergence results of this subsection.

THEOREM 5. *Let x be a solution of the integral equation (3.2) such that x_0 is regular (or, more generally, such that x_t is nondecreasing for all t). There exists a function $\tilde{y} \in \overline{F}$ such that $\lim_{t \rightarrow \infty} x_t(\alpha) = \tilde{y}(\alpha)$ holds for almost all α . Moreover, the set of nondecreasing fixed points contains F and is contained in \overline{F} .*

3.3. Stability and intercluster distances. As in the discrete case, we call *clusters* the discrete opinion values held by a positive measure set of agents at a fixed point \tilde{s} . For a cluster A , we denote by W_A , referred to as the *weight of the cluster*, the length of the interval $\tilde{s}^{-1}(A)$. By an abuse of language, we also call a cluster the interval $\tilde{s}^{-1}(A)$ of indices of the associated agents. In this section, we show that, for regular initial conditions, the limit to which the system converges satisfies a condition on the intercluster distance similar to the one in Theorem 3. From this result, we extract a necessary condition for stability of a fixed point.

THEOREM 6. *Let $\tilde{x}_0 \in X$ be an initial opinion function, let x be the solution of the integral equation (3.2), and let $\tilde{s} = \lim_{t \rightarrow \infty} x_t$ a.e. be the function to which x converges. If \tilde{x}_0 is regular, then*

$$(3.12) \quad |B - A| \geq 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}$$

holds for any two clusters A and B of \tilde{s} . As a consequence, \tilde{s} belongs to F and is thus a fixed point.

Proof. The idea of the proof is to rely on the continuity of x_t (as a function of α) at each t to guarantee the presence of perturbing agents between the clusters. Then, if (3.12) is violated, these perturbing agents will cause a merging of clusters.

Let A, B be two clusters of \tilde{s} , with $A < B$. Since $\tilde{s} \in \overline{F}$, $1 \leq B - A$. Let $m = \frac{W_A A + W_B B}{W_A + W_B}$ be their center of mass. Condition (3.12) is equivalent to requiring the center of mass to be at least unit distance away from at least one of the clusters. Suppose, to obtain a contradiction, that this condition is not satisfied, that is, that m is less than unit distance away from each of the two clusters A and B .

Since clusters are at least one unit apart, A and B are necessarily adjacent, and since (3.12) is violated, $B - A < 2$. From the monotonicity of x_t , there exists some $c \in I$ such that

$$\sup \left\{ \alpha : \lim_{t \rightarrow \infty} x_t(\alpha) = A \right\} = c = \inf \left\{ \alpha : \lim_{t \rightarrow \infty} x_t(\alpha) = B \right\}.$$

Moreover, we have the inclusions

$$(3.13) \quad \begin{aligned} (c - W_A, c) &\subseteq \left\{ \alpha : \lim_{t \rightarrow \infty} x_t(\alpha) = A \right\} \subseteq [c - W_A, c], \\ (c, c + W_B) &\subseteq \left\{ \alpha : \lim_{t \rightarrow \infty} x_t(\alpha) = B \right\} \subseteq [c, c + W_B]. \end{aligned}$$

Let us fix an $\epsilon > 0$. Since $x_t(\alpha)$ converges to $\tilde{s}(\alpha)$ for almost every α , since all x_t are nondecreasing, and since clusters are separated by a distance of at least 1, there exists a $t' > 0$ such that, for all $t \geq t'$, the following implications are satisfied:

$$(3.14) \quad \begin{aligned} \alpha < c - W_A - \epsilon &\Rightarrow x_t(\alpha) \leq A - 1, \\ \alpha \in (c - W_A + \epsilon, c - \epsilon) &\Rightarrow x_t(\alpha) \in (A - \epsilon, A + \epsilon), \\ \alpha \in (c + \epsilon, c + W_B - \epsilon) &\Rightarrow x_t(\alpha) \in (B - \epsilon, B + \epsilon), \\ \alpha > c + W_B + \epsilon &\Rightarrow x_t(\alpha) \geq B + 1. \end{aligned}$$

We introduce some new notation. With each function $\tilde{x} \in X$ we associate the function $\hat{l}_{\tilde{x}} : \mathfrak{R} \rightarrow (-1, 1)$ defined by

$$\hat{l}_{\tilde{x}}(q) = \int_{\tilde{x}^{-1}((q-1, q+1))} (\tilde{x}(\beta) - q) d\beta.$$

The value $\hat{l}_{\tilde{x}}(q)$ represents the derivative of the opinion of an agent whose current opinion is q . In particular, the differential equation (3.1) can be rewritten as $\frac{d}{dt}x_t(\alpha) = \hat{l}_{x_t}(x_t(\alpha))$.

Let us evaluate $\hat{l}_{x_t}(q)$ for $q \in [B - 1 + \epsilon, A + 1 - \epsilon]$. (Note that this interval is nonempty, because $B - A < 2$.) Observe first that $q - 1 \geq A - 1 + \epsilon > A - 1$, because $B - A \geq 1$. From the first relation in (3.14) and the continuity of x_t with respect to α , we obtain

$$x_t(c - W_A - \epsilon) \leq A - 1 < q - 1.$$

Observe also that $q - 1 \leq A - \epsilon$. From the second relation in (3.14) and the continuity of x_t , we obtain

$$x_t(c - W_A + \epsilon) \geq A - \epsilon \geq q - 1.$$

A similar argument around $q + 1$ shows that

$$x_t(c + W_B - \epsilon) \leq q + 1 < B + 1 \leq x_t(c + W_B + \epsilon).$$

Provided that ϵ is sufficiently small, these inequalities and the monotonicity of x_t imply that

$$[c - W_A + \epsilon, c + W_B - \epsilon] \subseteq x_t^{-1}((q - 1, q + 1)) \subseteq [c - W_A - \epsilon, c + W_B + \epsilon].$$

It also follows from the inclusions (3.14) that $\int_{c - W_A + \epsilon}^{c - \epsilon} (x_t(\beta) - q) d\beta = W_A(A - q) + O(\epsilon)$ and $\int_{c + \epsilon}^{c + W_B - \epsilon} (x_t(\beta) - q) d\beta = W_B(B - q) + O(\epsilon)$. Therefore,

$$\hat{l}_{x_t}(q) = W_A(A - q) + W_B(B - q) + O(\epsilon) = (W_A + W_B)(m - q) + O(\epsilon).$$

Observe now that, since the two clusters do not satisfy condition (3.12), their center of mass m lies in $(B - 1, A + 1)$. Provided that ϵ is sufficiently small, we have $m \in (B - 1 + \epsilon, A + 1 - \epsilon)$, and therefore $\hat{l}_{x_t}(B - 1 + \epsilon) > 0$ and $\hat{l}_{x_t}(A + 1 - \epsilon) < 0$ for all $t \geq t'$.

Recall that $\tilde{x}_0 \in X_m^M$ for some $m, M > 0$. From Theorem 4, x satisfies the differential equation (3.1) $\frac{d}{dt}x_t(\alpha) = \mathcal{L}(x_t)(\alpha) = \hat{l}_{x_t}(x_t(\alpha))$ and also condition (3.7). In particular, x_t is continuous and increasing with respect to $\alpha \in I$ for each t . There

exists therefore a positive length interval J such that $x_{t'}(J) \subseteq [B - 1 + \epsilon, A + 1 - \epsilon]$. Since $\hat{l}_{x_t}(B - 1 + \epsilon) > 0$ and $\hat{l}_{x_t}(A + 1 - \epsilon) < 0$ hold for any $t \geq t'$, and since $\frac{d}{dt}x_t(\alpha) = \hat{l}_{x_t}(x_t(\alpha))$, this implies that $x_t(J) \subseteq [B - 1 + \epsilon, A + 1 - \epsilon]$ for all $t \geq t'$. Since J has positive length, this contradicts the inclusions (3.13) on the convergence to the clusters A and B . \square

We note that the above proof also applies to any solution of (3.2) for which x_t is continuous with respect to $\alpha \in I$ for all t .

The following inclusions summarize the relations between the different sets of functions that have been proved above:

$$\begin{aligned} \left\{ \begin{array}{l} \lim_{t \rightarrow \infty} x_t : \\ x_0 \text{ regular} \end{array} \right\} &\subseteq \left\{ \begin{array}{l} s \in \overline{F} : \\ \text{satis. (3.12)} \end{array} \right\} \subset F \\ &\subseteq \text{fixed points} \subset \left\{ \begin{array}{l} \lim_{t \rightarrow \infty} x_t : \\ x_0 \text{ nondecreasing} \end{array} \right\} \subseteq \overline{F}. \end{aligned}$$

Observe that the set of possible $\lim_{t \rightarrow \infty} x_t$ strictly contains the set of fixed points. Every fixed point is indeed the limit of a system taking the fixed point itself as initial condition. On the other hand, a function such as $\tilde{x}_0(\alpha) = -1/2$ if $\alpha \in [0, \frac{1}{2}]$, and $\tilde{x}_0(\alpha) = 1/2$ otherwise, is a limit point, because (3.2) admits $x_t = \tilde{x}_0$ for all t as a solution, but it is not a fixed point, because (3.2) also admits other solutions with the same initial condition.

From Theorem 6, we can deduce a necessary condition for the stability of a fixed point under a classical definition of stability (in contrast to the nonstandard stability notion introduced for the discrete-agent system). Let \tilde{s} be a fixed point of (3.2). We say that \tilde{s} is stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\|\tilde{s} - \tilde{x}_0\|_1 \leq \delta$, then $\|\tilde{s} - x_t\|_1 \leq \epsilon$ for every t and every solution x of the integral equation (3.2) with \tilde{x}_0 as initial condition. It can be shown that this classical notion of stability is stronger than the stability with respect to the addition of a perturbing agent used in section 2.2. More precisely, if we view the discrete-agent system as a special case of the continuum model, stability under the current definition implies stability with respect to the definition used in section 2.2.

COROLLARY 1. *Let \tilde{s} be a fixed point of (3.2). If \tilde{s} is stable, then, for any two clusters A and B ,*

$$(3.15) \quad |B - A| \geq 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}.$$

Proof. Suppose that \tilde{s} does not satisfy this condition, and let K be the infimum of $\|\tilde{s} - \tilde{s}'\|_1$ over all $\tilde{s}' \in \overline{F}$ satisfying the condition. Clearly, $K > 0$. For every $\delta > 0$, there exist $M \geq m > 0$ and $\tilde{x}_0 \in X_m^M$ such that $\|\tilde{s} - \tilde{x}_0\|_1 \leq \delta$. Let x be the solution of the integral equation (3.2) with \tilde{x}_0 as initial condition, and let s' be the a.e.-limit of x_t . It follows from Theorem 6 that s' satisfies condition (3.12) and therefore that $\|\tilde{s} - s'\|_1 \geq K$. Using the dominated convergence theorem, we obtain $\lim_{t \rightarrow \infty} \|\tilde{s} - x_t\|_1 = \|\tilde{s} - s'\|_1$. As a result, $\lim_{t \rightarrow \infty} \|\tilde{s} - x_t\|_1 \geq K > 0$ holds for initial conditions \tilde{x}_0 arbitrarily close to \tilde{s} , and \tilde{s} is therefore unstable. \square

It is possible to prove that the strict inequality version of condition (3.15) is also necessary for stability. The proof for the case of equality relies on modifying the positions of an appropriate set of agents and “creating” some perturbing agents at the weighted average of the two clusters. See Chapter 10 of [9] or Theorem 6 in [3] for

the same proof applied to Krause’s model. We conjecture that the strict inequality version of condition (3.15) is also sufficient.

CONJECTURE 2. *A fixed point \tilde{s} of (3.2) is stable with respect to the norm $\|\cdot\|_1$ if and only if, for any two clusters A, B ,*

$$|B - A| > 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}.$$

We note that Conjecture 2 is a fairly strong statement. It implies, for example, that multiple clusters are indeed possible starting from regular initial conditions, which is an open question at present.

4. Relation between the discrete and continuum-agent models. We now formally establish a connection between the discrete-agent and the continuum-agent models and use this connection to argue that the validity of Conjecture 2 implies the validity of Conjecture 1 in section 2.2. Toward this purpose, we begin by proving a result on the continuity of the opinion evolution with respect to the initial conditions.

PROPOSITION 4. *Let x be the solution of the continuum model (3.2) for some regular initial condition $\tilde{x}_0 \in X_m^M$. For every $\epsilon > 0$ and $T > 0$, there exists a $\delta > 0$ such if y is a solution of the continuum model (3.2) and $\|y_0 - \tilde{x}_0\|_\infty \leq \delta$, then $\|y_t - x_t\|_\infty \leq \epsilon$ for all $t \in [0, T]$.*

Proof. From Theorem 4, $x_t \in X_{me^{-t}}$ for all t . Lemma 1 then implies that, for any $\tilde{y}_t \in Y$ and any $t \in [0, T]$,

$$(4.1) \quad \|\mathcal{L}(\tilde{y}_t) - \mathcal{L}(x_t)\|_\infty \leq \left(2 + \frac{8}{m}e^t\right) \|\tilde{y}_t - x_t\|_\infty \leq \left(2 + \frac{8}{m}e^T\right) \|\tilde{y}_t - x_t\|_\infty.$$

For every $\alpha \in I$, we have

$$y_t(\alpha) - x_t(\alpha) = y_0(\alpha) - \tilde{x}_0(\alpha) + \int_0^t (\mathcal{L}(y_\tau)(\alpha) - \mathcal{L}(x_\tau)(\alpha)) d\tau.$$

It follows from this relation and from the bound (4.1) that

$$\|y_t - x_t\|_\infty - \|y_s - x_s\|_\infty \leq \int_s^t \left(2 + \frac{8}{m}e^\tau\right) \|y_\tau - x_\tau\|_\infty d\tau$$

holds for any $0 \leq s \leq t \leq T$. This implies that, for all $t \in [0, T]$,

$$\|y_t - x_t\|_\infty \leq \|y_0 - \tilde{x}_0\|_\infty e^{t(2 + \frac{8}{m}e^T)} \leq \|y_0 - \tilde{x}_0\|_\infty e^{T(2 + \frac{8}{m}e^T)}.$$

Let us now fix an $\epsilon > 0$ and take $\delta > 0$ such that $\delta e^{T(2 + \frac{8}{m}e^T)} \leq \epsilon$. It follows from the inequality above that if $\|\tilde{y}_0 - \tilde{x}_0\|_\infty \leq \delta$, then $\|y_t - x_t\|_\infty \leq \epsilon$ for every $t \in [0, T]$. \square

The following result shows that the continuum-agent model can be interpreted as the limit when $n \rightarrow \infty$ of the discrete-agent model on any time interval of finite length. To avoid any risk of ambiguity, we use ξ to denote discrete vectors in what follows. Moreover, we assume that such vectors are always sorted (i.e., $j > i \Rightarrow \xi_j \geq \xi_i$). We define the operator G that maps a discrete (nondecreasing) vector to a function by $G(\xi)(\alpha) = \xi_i$ if $\alpha \in [\frac{i-1}{n}, \frac{i}{n})$, and $G(\xi)(1) = \xi(n)$, where n is the dimension of the vector ξ . Let ξ be a solution of the discrete-agent model (2.1) with initial condition $\xi(0)$. One can verify that $G(\xi(t))$ is a solution to the continuum-agent

integral equation (3.2) with $G(\xi(0))$ as initial condition. As a result, the discrete-agent model can be simulated by the continuum-agent model. The next proposition provides a converse, in some sense, over finite-length time intervals.

THEOREM 7. *Consider a regular initial opinion function \tilde{x}_0 , and let $(\xi^{(n)})_{n>0}$ be a sequence of (nondecreasing) vectors in \mathfrak{R}^n such that $\lim_{n \rightarrow \infty} \|G(\xi^{(n)}(0)) - \tilde{x}_0\|_\infty = 0$, and such that, for each n , $\xi^{(n)}(0)$ is a proper initial condition, admitting a unique solution $\xi^{(n)}(t)$. Then, for every T and every $\epsilon > 0$, there exists n' such that*

$$\left\| G(\xi^{(n)}(t)) - x_t \right\|_\infty \leq \epsilon$$

holds for all $t \in [0, T]$ and $n \geq n'$.

Proof. The result follows directly from Proposition 4 and from the fact that $G(\xi^{(n)}(t))$ is a solution of (3.2) with the initial condition $G(\xi^{(n)}(0))$. \square

When \tilde{x} is regular, a simple way of building such a sequence $(\xi^{(n)}(0))_{n>0}$ is to take $\xi_i^{(n)}(0) = \tilde{x}_0(i/n)$. Theorem 7 implies that the discrete-agent model approximates arbitrarily well the continuum model for arbitrarily large periods of time, provided that the initial distribution of discrete opinions approximates sufficiently well the initial conditions of the continuum model. Now recall that, according to Theorem 6 and for regular initial conditions, the continuum-agent model converges to a fixed point satisfying the intercluster distance condition (3.12). The conjunction of these two results thus seems to support our Conjecture 1, that the discrete-agent model converges to an equilibrium satisfying this same condition, provided that the number of agents is sufficiently large and that their initial opinions approximate some regular function. This argument, however, is incomplete, because the approximation result in Theorem 7 is valid only over finite, not infinite, time intervals. Nevertheless, we will now show that this reasoning is valid, with rare exceptions, if Conjecture 2 holds.

PROPOSITION 5. *Suppose that \tilde{x}_0 is regular, and suppose that the limit \tilde{s} of the resulting solution x of (3.2) is stable and its clusters satisfy*

$$(4.2) \quad |B - A| > 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}.$$

Let $\xi(0) \in \mathfrak{R}^n$ be a vector whose n entries are randomly and independently selected according to a probability density function corresponding to \tilde{x}_0 . Then, the clusters of the limit of the corresponding solution of (2.1) satisfy (4.2), with probability that tends to 1 as $n \rightarrow \infty$.

Proof. Let $\tilde{s} = \lim_{t \rightarrow \infty} x_t$, which is assumed to be stable and to satisfy (4.2). Since (4.2) involves a strict inequality, we see that there exists some $K > 0$ such that the clusters of any fixed point s' that satisfies $\|s' - \tilde{s}\|_1 \leq K$ must also satisfy (4.2). Furthermore, since \tilde{s} is stable, there exists some $\epsilon > 0$ such that if a solution of the integral equation (3.2) satisfies $\|y_{t'} - \tilde{s}\|_1 < \epsilon$ for some t' , then $\|y_t - \tilde{s}\|_1 \leq K$ for all $t \geq t'$. To summarize, if a converging trajectory y_t comes within ϵ of \tilde{s} , that trajectory can converge only to a fixed point whose clusters satisfy (4.2).

Suppose now that $\xi(0)$ is a vector generated at random, as in the statement of the proposition, and whose components are reindexed so that they are nondecreasing. It follows from Kolmogorov–Smirnov theorems (see [8], for example) and the regularity of \tilde{x}_0 that, for any given $\delta > 0$, the probability of the event $\|G(\xi(0)) - \tilde{x}_0\|_\infty < \delta$ converges to 1 as $n \rightarrow \infty$.

Since x_t converges to \tilde{s} a.e., the dominated convergence theorem implies that there exists some t' such that $\|x_t - \tilde{s}\|_1 < \epsilon/2$. Now let $\xi(t)$ be a solution of (2.1) for

the initial condition $\xi(0)$, the existence of which is guaranteed with probability 1 by Theorem 1. Since $G(\xi(t))$ is also a solution of the problem (3.2) with initial condition $G(\xi(0))$, Proposition 4 implies that when δ is chosen sufficiently small (which happens with high probability when n is sufficiently large) we will have $\|G(\xi(t')) - x_{t'}\|_\infty < \epsilon/2$ and, consequently, $\|G(\xi(t')) - x_{t'}\|_1 < \epsilon/2$. Therefore, with probability that tends to 1 as n increases,

$$\|G(\xi(t')) - \tilde{s}\|_1 \leq \|G(\xi(t')) - x_{t'}\|_1 + \|x_{t'} - \tilde{s}\|_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that, with probability that tends to 1 as n increases, the limit $G(\xi(t))$ is a fixed point that satisfies (4.2). \square

We now use Proposition 5 to establish the connection between our two conjectures. Suppose that Conjecture 2 holds. Let \tilde{x}_0 be a regular initial condition. By Theorem 6, the resulting trajectory converges to a fixed point \tilde{s} that satisfies the nonstrict inequality (3.12). We expect that generically the inequality will actually be strict, in which case, according to Conjecture 2, \tilde{s} is stable. Therefore, subject to the genericity qualification above, Proposition 5 implies the validity of Conjecture 1.

5. Conclusions. We have analyzed a simple continuous-time multiagent system for which the interaction topology depends on the agent states. We worked with the explicit dynamics of the interaction topology, which raised a number of difficulties, because the resulting system is highly nonlinear and discontinuous. This is in contrast to the case of exogenously determined topology dynamics, which result in time-varying but linear dynamics.

After establishing convergence to a set of clusters in which agents share the same opinion, we focused on the intercluster distances. We proposed an explanation for the experimentally observed distances based on a notion of stability that is tailored to our context. This also led us to conjecture that the probability of convergence to a stable equilibrium (in which certain minimal intercluster distances are respected) tends to 1 as the number of agents increases.

We then introduced a variant of the model, involving a continuum of agents. For regular initial conditions, we proved the existence and uniqueness of solutions, the convergence of the solution to a set of clusters, and a nontrivial bound on the intercluster distances of the same form as the necessary and sufficient condition for stability for the discrete-agent model. Finally, we established a link between the discrete and continuum models and related our first conjecture to a seemingly simpler conjecture.

The results presented here are parallel to, but much stronger than, those that we obtained for Krause's model of opinion dynamics [3]. Indeed, we have provided here a full analysis of the continuum model under the mild and easily checkable assumption of regular initial conditions.

The tractability of the model in this paper can be attributed to (i) the inherent symmetry of the model, and (ii) the fact that it runs in continuous time, although the latter aspect also raised nontrivial questions related to the existence and uniqueness of solutions. We note, however, that similar behaviors have also been observed for systems without such symmetry. One can therefore wonder whether the symmetry is really necessary, or just allows for comparatively simpler proofs. One can similarly wonder whether our results admit counterparts in models involving high-dimensional opinion vectors, where one can no longer rely on monotonic opinion functions and order-preservation results.

As in our work on Krause's model, our study of the system on a continuum and the distances between the resulting clusters uses the fact that the density of agents between the clusters that are being formed is positive at any finite time. This, however, implies that, unlike the discrete-agent case, the clusters always remain indirectly connected, and it is not clear whether this permanent connection can eventually force clusters to merge. In fact, it is an open question whether there exists a regular initial condition that leads to multiple clusters, although we strongly suspect this to be the case. A simple proof would consist of an example of regular initial conditions that admit a closed-form formula for x_t . However, this is difficult, because of the discontinuous dynamics. The only available examples of this type converge to a single cluster, as, for example, in the case of any two-dimensional distribution of opinions with circular symmetry (see [5]).

Appendix A. Existence and uniqueness of solutions to the discrete-agent equation: Proof of Theorem 1. We sketch here the proof of Theorem 1, a full version of which is available in [4]. Observe first that if x is the unique solution of the system (2.1) for a given initial condition, then $x_i(t) = x_j(t)$ implies that $x_i(t') = x_j(t')$ holds for all $t' > t$. Indeed, one could otherwise build another solution by switching x_i and x_j after the time t , in contradiction with the uniqueness of the solution. Therefore, every initial condition \tilde{x}_0 satisfying condition (a) in the definition of proper initial conditions automatically satisfies condition (c).

Let us now fix the number of agents n and, for every undirected graph G on n vertices, with edge set E , define $X_G \subseteq \mathfrak{R}^n$ as the subset in which $|x_i - x_j| < 1$ if $(i, j) \in E$, and $|x_i - x_j| > 1$ if $(i, j) \notin E$. When restricted to X_G , the system (2.1) becomes the linear time-invariant differential system

$$(A.1) \quad \dot{x}_i = \sum_{j:(i,j) \in E} (x_j - x_i),$$

which admits a unique solution for any initial condition. This system can be more compactly written as $\dot{x} = -L_G x$, where L_G is the Laplacian matrix of the graph G .

Consider an initial condition $\tilde{x} \in \mathfrak{R}^n$, and suppose that $\tilde{x} \in X_{G_0}$ for some G_0 . Let x_{G_0} be the unique solution of $\dot{x} = -L_{G_0} x$ with $x_{G_0}(0) = \tilde{x}$. If this solution always remains in X_{G_0} , it is necessarily the unique solution of (2.1). Otherwise, let $t_1 > 0$ be the first time at which $x_{G_0}(t) \in \partial X_{G_0}$, and set $x(t) = x_{G_0}(t)$ for all $t \in [0, t_1]$. By the definition of the sets X_G , the point $x(t_1)$ also belongs to the boundary of at least one other set X_{G_1} , with G_1 and G_0 differing only by one edge (i, j) . We consider here the case where $(i, j) \in E_0$, $(i, j) \notin E_1$, and $x_i(t_1) > x_j(t_1)$, but a similar argument can be made in the three other possibilities. We also assume that $x(t_1)$ belongs to the closure of no other set X_G and that $(L_{G_0} x(t_1))_i - (L_{G_0} x(t_1))_j \neq 0$. This assumption does not always hold, but can be proved to hold for all boundary points that can be reached, except for a set that has zero measure (with respect to the relative Lebesgue measure defined on the lower-dimensional boundary).

Since $x_i(t) - x_j(t) < 1$ for t just before t_1 and $x_i(t_1) - x_j(t_1) = 1$, there must hold that $\lim_{t \uparrow t_1} (\dot{x}_i(t) - \dot{x}_j(t)) \geq 0$, and thus $-(L_{G_0} x(t_1))_i + (L_{G_0} x(t_1))_j > 0$, because we have assumed that the latter quantity is nonzero. Recall that G_1 is obtained from G_0 by removing the edge (i, j) . Since $x_i(t_1) - x_j(t_1) = 1$, we have

$$\begin{aligned} -(L_{G_1} x)_i + (L_{G_1} x)_j &= -(L_{G_0} x)_i + (L_{G_0} x)_j - 2(x_j(t_1) - x_i(t_1)) \\ &= -(L_{G_0} x)_i + (L_{G_0} x)_j + 2 \\ &> 0. \end{aligned}$$

So, if the solution x can be extended after t_1 , there must hold that $x_i(t) - x_j(t) > 1$ for all t in some positive length open interval starting at t_1 . This implies that $x(t) \in X_{G_1}$ on some (possibly smaller) positive length open interval starting at t_1 , because $x(t_1)$ is at a positive distance from all sets X_G other than X_{G_0} and X_{G_1} . On this latter interval, any solution x must thus satisfy $\dot{x} = -L_{G_1}x$. This linear system admits a unique solution x_{G_1} for which $x_{G_1}(t_1) = x(t_1)$. Moreover, the solution remains in X_{G_1} for some positive length time period, again because $-(L_{G_1}x)_i + (L_{G_1}x)_j > 0$ and because $x(t_1)$ is at a positive distance from all sets X_G other than X_{G_0} and X_{G_1} . If it remains in X_{G_1} forever, we extend x by setting $x(t) = x_{G_1}(t)$ on $[t_1, \infty)$. Otherwise, we extend x , as before, on the interval $[t_1, t_2]$, where t_2 is the first time after t_1 at which $x_{G_1} \in \partial X_{G_1}$. In both cases, x is a solution to (2.1), on $[0, \infty)$ or $[0, t_2]$, respectively, and is unique. Indeed, we have seen that it is the unique solution on $[0, t_1)$, that any extended solution should then enter X_{G_1} , and that there is a unique solution entering X_{G_1} at t_1 via $x(t_1)$.

One can prove that, for almost all initial conditions, this process can be continued recursively without encountering any “problematic boundary points,” namely, those for which $(L_G x)_i - (L_G x)_j = 0$, or those incident to more than two sets.⁷ Such a recursive construction ends after a finite number of transitions if a solution eventually enters and remains forever in a set X_G . In this case, we have proved the existence of a unique solution (2.1) on \mathbb{R}^+ , differentiable everywhere but on a finite set of times. Alternatively, the construction may result in an infinite sequence of transition times t_1, t_2, \dots . If this sequence diverges, we again have a unique solution. A problem arises only if this sequence converges to some finite time T^* , in which case we could establish existence and uniqueness only on $[0, T^*)$. The following lemma shows that this problematic behavior will not arise and concludes the proof of Theorem 1.

LEMMA 4. *Suppose that the above recursive construction never encounters problematic boundary points (in the sense defined above) and produces an infinite sequence of transition times t_0, t_1, \dots . Then, this sequence diverges, and therefore there exists a unique solution x , defined for all $t \geq 0$. Moreover, $\dot{x}(t)$ does not converge to 0 when $t \rightarrow \infty$.*

Proof. Since the sequence t_1, t_2, \dots of transition times is infinite, a nonempty set of agents is involved in an infinite number of transitions, and there exists a time T after which every agent involved in a transition will also be involved in a subsequent one. Consider now a transition occurring at $s_1 > T$ and involving agents i and j . We denote by $\dot{x}_i(s_1^-)$ and $\dot{x}_i(s_1^+)$ the limits $\lim_{t \uparrow s_1} \dot{x}_i(t)$ and $\lim_{t \downarrow s_1} \dot{x}_i(t)$, respectively. (Note that these limits exist, because, away from boundary points, the function x is continuously differentiable.)

Suppose without loss of generality that $x_i > x_j$. If i and j are connected before s_1 but not after, then the update equation (2.1) implies that $\dot{x}_i(s_1^+) = \dot{x}_i(s_1^-) - (x_j(s_1) - x_i(s_1))$. Noting that $x_i(s_1) - x_j(s_1) = 1$, we conclude that $\dot{x}_i(s_1^+) = \dot{x}_i(s_1^-) + 1$. Moreover, $x_i - x_j$ must have been increasing just before s_1 , so that $\dot{x}_i(s_1^-) \geq \dot{x}_j(s_1^-)$. If, on the other hand, i and j are connected after s_1 but not before, then $\dot{x}_j(s_1^+) = \dot{x}_j(s_1^-) + 1$, and since $x_i - x_j$ must have been decreasing just before s_1 , there holds that $\dot{x}_j(s_1^-) \geq \dot{x}_i(s_1^-)$. In either case, there exists an agent $k_1 \in \{i, j\}$ for which $\dot{x}_{k_1}(s_1^+) = \max\{\dot{x}_i(s_1^-), \dot{x}_j(s_1^-)\} + 1$. It follows from $s_1 > T$ that this agent will get involved in some other transition at a further time. Call s_2 the first such time.

The definition (2.1) of the system implies that, in between transitions, $|\dot{x}_i(t)| \leq n$

⁷The authors are pleased to acknowledge discussions with Prof. Eduardo Sontag on this assertion and its proof.

for all agents. Using (2.1) again, this implies that $|\ddot{x}_i(t)| \leq 2n^2$ for all t at which i is not involved in a transition. Therefore, $\dot{x}_{k_1}(s_2^-) \geq \dot{x}_{k_1}(s_1^+) - 2n^2(s_2 - s_1) = x_{k_1}(s_1^-) + 1 - 2n^2(s_2 - s_1)$. Moreover, by the same argument as above, there exists a k_2 for which $\dot{x}_{k_2}(s_2^+) = \dot{x}_{k_1}(s_2^-) + 1 \geq x_{k_1}(s_1^-) + 2 - 2n^2(s_2 - s_1)$. Continuing recursively, we can build an infinite sequence of transition times s_1, s_2, \dots (a subsequence of t_1, t_2, \dots) such that, for every m ,

$$\dot{x}_{k_m}(s_m^+) \geq \dot{x}_i(s_1^-) + m - 2n^2(s_m - s_1)$$

holds for some agent k_m . Since all velocities are bounded by n , this implies that $s_m - s_1$ must diverge as m grows and therefore that the sequence t_1, t_2, \dots of transition times diverges. \square

Note that the proof of Lemma 4 provides an explicit bound on the number of transitions that can take place during any given time interval.

Appendix B. Existence and uniqueness of solutions to the continuum-agent model: Proof of Theorem 4. Let us fix a function $\tilde{x}_0 \in X_m^M$, with $0 < m \leq M$. Let us also fix some t_1 such that

$$(B.1) \quad \frac{m}{2} \leq m - 2Mt \leq me^{-t} \leq Me^{4t/m} \leq M + \frac{8}{m}t \leq 2M$$

and

$$(B.2) \quad \left(2 + \frac{16}{m}\right)t \leq \frac{1}{2}, \quad e^t \leq 2,$$

for all $t \in [0, t_1]$. We note, for future reference, that t_1 can be chosen as a function $f(m, M)$, where f is continuous and positive.

Recall that we defined the operator G that maps measurable functions $x : I \times [0, t_1] \rightarrow \mathfrak{R}$ into the set of such functions by

$$(G(x))_t(\alpha) = \tilde{x}_0(\alpha) + \int_0^t \mathcal{L}(x_\tau)(\alpha) d\tau.$$

Observe that x is a solution of the integral equation (3.2) if and only if $x_0 = \tilde{x}_0$ and $x = G(x)$. Let P be the set of measurable functions $x : I \times [0, t_1] \rightarrow \mathfrak{R} : (\alpha, t) \rightarrow x_t(\alpha)$ such that $x_0 = \tilde{x}_0$ and such that, for all $t \in [0, t_1]$, we have $x_t \in X_{m-2Mt}^{M+8(M/m)t}$ or, in detail,

$$(B.3) \quad m - 2Mt \leq \frac{x_t(\beta) - x_t(\alpha)}{\beta - \alpha} \leq M + 8\frac{M}{m}t$$

for all $t \in [0, t_1]$ and all $\beta \neq \alpha$. In particular, if $x \in P$, then

$$(B.4) \quad \frac{m}{2} \leq \frac{x_t(\beta) - x_t(\alpha)}{\beta - \alpha} \leq 2M$$

for all $t \in [0, t_1]$ and all $\beta \neq \alpha$.

We note that P , endowed with the $\|\cdot\|_\infty$ norm, defined by

$$\|x\|_\infty = \max_{\alpha \in I, t \in [0, t_1]} |x_t(\alpha)|,$$

is a complete metric space. We will apply Banach’s fixed point theorem to the operator G on P . The first step is to show a contraction property of G .

LEMMA 5. *The operator G is contracting on P . In particular, $\|G(y) - G(x)\|_\infty < \frac{1}{2} \|y - x\|_\infty$ for all $x, y \in P$.*

Proof. Let $x, y \in P$. For any $t \in [0, t_1]$, we have $x_t \in X_{m/2}$ (cf. (B.4)), and Lemma 1 implies that

$$\|\mathcal{L}(y_t) - \mathcal{L}(x_t)\|_\infty \leq \left(2 + \frac{16}{m}\right) \|y_t - x_t\|_\infty.$$

Then, for every $\alpha \in I$,

$$\begin{aligned} |(G(y))_t(\alpha) - (G(x))_t(\alpha)| &= \left| \int_0^t (\mathcal{L}(y_\tau)(\alpha) - \mathcal{L}(x_\tau)(\alpha)) \, d\tau \right| \\ &\leq \int_0^t \|\mathcal{L}(y_\tau) - \mathcal{L}(x_\tau)\|_\infty \, d\tau \\ &\leq \int_0^t \left(2 + \frac{16}{m}\right) \|y_\tau - x_\tau\|_\infty \, d\tau \\ &\leq \left(2 + \frac{16}{m}\right) t \|y - x\|_\infty \\ &\leq \frac{1}{2} \|y - x\|_\infty, \end{aligned}$$

where the last inequality follows from (B.2). □

Before applying Banach’s fixed point theorem, we also need to verify that G maps P into itself.

LEMMA 6. *If $x \in P$, then $G(x) \in P$.*

Proof. Suppose that $x \in P$. By definition, $G(x)_0 = \tilde{x}_0$, and we need only prove that $G(x)$ satisfies condition (B.3). For $t \in [0, t_1]$ and $\alpha \leq \beta$, we have

$$G(x)_t(\beta) - G(x)_t(\alpha) = \tilde{x}_0(\beta) - \tilde{x}_0(\alpha) + \int_0^t (\mathcal{L}(x_\tau)(\beta) - \mathcal{L}(x_\tau)(\alpha)) \, d\tau.$$

It follows from the first part of Lemma 2 and from (B.4) that

$$\mathcal{L}(x_\tau)(\beta) - \mathcal{L}(x_\tau)(\alpha) \geq -(x_\tau(\beta) - x_\tau(\alpha)) \geq -2M(\beta - \alpha).$$

Since $\tilde{x}_0(\beta) - \tilde{x}_0(\alpha) \geq m(\beta - \alpha)$, for any $t \in [0, t_1]$, we have

$$G(x)_t(\beta) - G(x)_t(\alpha) \geq m(\beta - \alpha) - \int_0^t 2M(\beta - \alpha) \, d\tau = (m - 2Mt)(\beta - \alpha),$$

so that $G(x)$ satisfies the first inequality in (B.3).

We now use the second part of Lemma 2 and (B.4) to obtain

$$\mathcal{L}(x_\tau)(\beta) - \mathcal{L}(x_\tau)(\alpha) \leq \frac{4}{m} (x_\tau(\beta) - x_\tau(\alpha)) \leq 8\frac{M}{m}(\beta - \alpha).$$

Since $\tilde{x}_0(\beta) - \tilde{x}_0(\alpha) \leq M(\beta - \alpha)$, for any $t \in [0, t_1]$, we have

$$G(x)_t(\beta) - G(x)_t(\alpha) \leq M(\beta - \alpha) + \int_0^t 8\frac{M}{m}(\beta - \alpha) \, d\tau = \left(M + 8\frac{M}{m}t\right) (\beta - \alpha).$$

Therefore $G(x)$ also satisfies the second inequality in (B.3) and belongs to P . \square

By Lemmas 5 and 6, G maps P into itself and is a contraction. It follows, from the Banach fixed point theorem, that there exists some unique $x^* \in P$ such that $x^* = G(x^*)$. We now show that no other fixed point can be found outside P .

LEMMA 7. *If a measurable function $x : I \times [0, t_1)$ satisfies $x = G(x)$, then it satisfies condition (3.7) and, in particular, $x \in P$.*

Proof. Suppose that the function $x : I \times [0, t_1] \rightarrow \mathfrak{R} : (\alpha, t) \rightarrow x_t(\alpha)$ satisfies $x = G(x)$; that is, $x_t(\alpha) = \tilde{x}_0(\alpha) + \int_0^t \mathcal{L}(x_\tau)(\alpha) d\tau$ for all t and $\alpha \in I$. It follows from the first part of Lemma 2 that

$$(B.5) \quad (x_t(\beta) - x_t(\alpha)) - (x_0(\beta) - x_0(\alpha)) \geq - \int_0^t (x_\tau(\beta) - x_\tau(\alpha)) d\tau$$

holds for $\alpha \leq \beta$ and $0 \leq s \leq t$. Since $h(t) - h(0) \geq - \int_0^t h(\tau) d\tau$ for all $t \geq 0$ implies that $h(t) \geq h(0)e^{-t}$ (special case of Gronwall's inequality), and since $\tilde{x}_0 \in X_m$ implies that $(x_0(\beta) - x_0(\alpha)) \geq m(\beta - \alpha)$, it follows from (B.5) that

$$x_t(\beta) - x_t(\alpha) \geq e^{-t} (x_0(\beta) - x_0(\alpha)) \geq me^{-t}(\beta - \alpha),$$

which proves the first inequality in (3.7) and also that $x_t \in X_{me^{-t}}$ for all t . Using this bound, it follows from the second part of Lemma 2 that

$$\begin{aligned} (x_t(\beta) - x_t(\alpha)) - (x_0(\beta) - x_0(\alpha)) &\leq \int_0^t \frac{2}{m} e^t (x_\tau(\beta) - x_\tau(\alpha)) d\tau \\ &\leq \frac{4}{m} \int_0^t (x_\tau(\beta) - x_\tau(\alpha)) d\tau, \end{aligned}$$

where the last inequality follows from (B.2). Therefore,

$$x_t(\beta) - x_t(\alpha) \leq e^{4t/m}(\tilde{x}_0(\beta) - \tilde{x}_0(\alpha)) \leq Me^{4t/m}(\beta - \alpha),$$

where we have used the assumption that $\tilde{x}_0 \in X^M$. This shows the second inequality in (B.3) and, in particular, that $x \in P$. \square

We have shown so far that the integral equation (3.2) has a unique solution x^* (for $t \in [0, t_1]$), which also belongs to P . We argue that it is also the unique solution to the differential equation (3.1). Since $\mathcal{L}(x_t^*)$ is bounded for all t , it follows from the equality

$$(B.6) \quad x_t^* = \tilde{x}_0 + \int_0^t \mathcal{L}(x_\tau^*) d\tau$$

that x_t^* is continuous with respect to t under the $\|\cdot\|_\infty$ norm; that is, there holds $\lim_{\tau \rightarrow t} \|x_\tau^* - x_t^*\|_\infty = 0$ for all t . By Lemma 7, x^* satisfies condition (3.7), and then Lemma 1 implies that \mathcal{L} is a Lipschitz continuous at every x_t^* . The continuity of x_t^* with respect to t then implies that $\mathcal{L}(x_t^*)$ also evolves continuously with t . Therefore, we can differentiate (B.6), to obtain $\frac{d}{dt} x_t^*(\alpha) = \mathcal{L}(x_t^*)(\alpha)$, for all t and α . The function x^* is thus a solution to the differential equation (3.1). Finally, since every solution of the differential equation is also a solution of the integral equation, which admits a unique solution, the solution of the differential equation is also unique.

To complete the proof of the theorem, it remains to show that the solution x^* can be extended to all $t \geq 0$. Let $\tilde{x}_1 = x_{t_1}^*$. It follows from (3.7) that $\tilde{x}_1 \in X_{m_1}^{M_1}$, with

$m_1 = me^{-t_1}$ and $M_1 = Me^{4t_1/m}$. By repeating the argument given for $[0, t_1]$, there exists a unique x^{**} , defined on $I \times [t_1, t_2]$, such that

$$x_t^{**} = \tilde{x}_1 + \int_{t_1}^t \mathcal{L}(x_\tau^{**}) d\tau$$

for all $t \in [t_1, t_2]$ and where $t_2 - t_1 = f(m_1, M_1)$. One can easily verify that the function obtained by concatenating x^* and x^{**} is a (unique) solution of the integral and differential equations on $[0, t_2]$ and that it satisfies the bound (3.7). Repeating this argument, we show the existence of a unique solution on every $[0, t_n]$, with

$$t_{n+1} - t_n = f(me^{-t_n}, Me^{4t_n/m}).$$

Since this recursion can be written as $t_{n+1} = t_n + g(t_n)$, with g continuous and positive on $[0, \infty)$, the sequence t_n diverges. (To see this, note that if $t_n \leq t^*$ for all n , then $g(t_n) \geq \min_{0 \leq t \leq t^*} g(t) > 0$ for all n , which proves that $t_n \rightarrow \infty$, a contradiction.) This completes the proof of Theorem 4.

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