Existence and uniqueness of solutions for a continuous-time opinion dynamics model with state-dependent connectivity

Vincent D. Blondel, Julien M. Hendrickx and John N. Tsitsiklis *

July 24, 2009

Abstract

We prove that the continuous-time discrete-agent system studied in [1] admits a unique solution for almost all initial conditions.

We consider the following continuous-time model: each of n agents, labeled $1, \ldots, n$, maintains a real number $x_i(t)$, which is a continuous function of time, and evolves according to

$$x_i(t) = x_i(0) + \int_0^t \sum_{j:|x_i(\tau)-x_j(\tau)|<1} (x_j(\tau) - x_i(\tau))\, d\tau. \tag{1}$$

We say that $\tilde{x} \in \Re^n$ is a *proper initial condition of (1)* if:
(a) There exists a unique $x : \Re^+ \to \Re^n : t \to x(t)$ satisfying (1), and such that $x(0) = \tilde{x}$.
(b) The subset of $\Re^+$ on which x is not differentiable is at most countable, and has no accumulation points.
(c) If $x_i(t) = x_j(t)$ holds for some t, then $x_i(t') = x_j(t')$, for every $t' \geq t$.
We then say that the solution x is a *proper solution* of (1). We provide in this note a detailed proof of the following theorem.

Theorem 1. *Almost all $\tilde{x} \in \Re^n$ are proper initial conditions, that is, the set of non-proper initial conditions has zero Lebesgue measure.*

1 Constructing a solution

Let us fix the number of agents n, and for each graph $G(\{1, \ldots, n\}, E)$ define $X_G \subseteq \Re^n$ as the subset in which $|x_i - x_j| < 1$ if $(i, j) \in E$, and $|x_i - x_j| > 1$ if $(i, j) \notin E$. (Note that X_G is an open set.) When restricted to any given X_G, our system (1) is equivalent to the linear time-invariant differential equation

$$\dot{x}_i = \sum_{j:(i,j)\in E} (x_j - x_i), \tag{2}$$

which admits a unique solution for any initial condition. This system can be more compactly rewritten as $\dot{x} = -L_G x$, where L_G is the Laplacian matrix of the graph G.

Consider an initial condition $\tilde{x}$ and suppose that it belongs to X_{G_0} for some G_0. We construct a solution to (1) as follows. First, let $x^0 : \Re^+ \to \Re^n$ be the unique solution of $\dot{x}(t) = -L_{G_0} x(t)$ for which

*This research was supported by the National Science Foundation under grant ECCS-0701623, by the Concerted Research Action (ARC) "Large Graphs and Networks" of the French Community of Belgium, and by the Belgian Programme on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its authors. Julien Hendrickx holds postdoctoral fellowships from the F.R.S.-FNRS (Belgian Fund for Scientific Research) and the B.A.E.F. (Belgian American Education Foundation). V. D. Blondel is with Department of Mathematical Engineering, Université catholique de Louvain, Avenue Georges Lemaitre 4, B-1348 Louvain-la-Neuve, Belgium; vincent.blondel@uclouvain.be J. M. Hendrickx and J. N. Tsitsiklis are with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139, USA; jm_hend@mit.edu, jnt@mit.edu. Part of this reasearch was conducted when J. M. Hendrickx was at the Université catholique de Louvain.

$x^0(0) = \tilde{x}$, and let $t_0 = 0$. Then, for successive values of k, starting with $k = 0$, and given t_k and $x(t_k)$, do the following:

If $x^k(t) \in X_{G_k}$ for all $t > t_k$, **then**
    Set $x(t) = x^k(t)$ for all $t > t_k$
    Stop (success)
else
    Let $t_{k+1} = \min_{t>t_k}\{t : x^k(t) \in \partial X_{G_k}\}$
    Set $x(t) = x^k(t)$ for all $t \in (t_k, t_{k+1}]$

    If $x(t_k)$ belongs to the boundary of more than two sets X_G
        Stop (failure)
    End if

    Let G_{k+1} be such that $x(t_k) \in \partial X_{G_k} \cap \partial X_{G_{k+1}}$
    Let x^{k+1} be the unique solution of $\dot{x}^{k+1} = -L_{G_{k+1}} x^{k+1}$ for which $x^{k+1}(t_{k+1}) = x(t_{k+1})$

    If there exists some $\epsilon > 0$ such that $x^{k+1}(t) \in X_{G_{k+1}}$ for all $t \in (t_{k+1}, t_{k+1} + \epsilon)$ **then**
        increment k and start the next iteration
    else Stop (failure)
End if

We say that the procedure fails if it stops on one of the two "stop" commands labeled with "(failure)", and succeeds otherwise. Observe that if it succeeds, it either produces (i) an increasing finite sequence $t_0, t_1, \ldots, t_M$, and a function x defined for all positive t, or (ii) an infinite increasing sequence $t_0, t_1, \ldots$, and a function x defined on the interval $[0, \lim_{k\to\infty} t_k)$. This interval will not cover the entire time axis if the sequence of t_k converges to a finite limit. In any case, one can verify that the function x produced is a solution to the system (1) on its domain of definition, $[0, \lim_{k\to\infty} t_k)$.

In the sequel, we use e_i to denote the vector in $\Re^n$ whose ith entry is equal to one, with all other entries equal to zero.

2 Problematic points

We now define a set of problematic points on the boundaries of the sets X_G and show that, provided that the procedure described above never encounters such a problematic point, it produces a proper solution. We say that a point $x \in \Re^n$ is *problematic* if it belongs to the boundary of more than two sets X_{G_i} or if the following three conditions are satisfied.

(a) $x \in \overline{X}_{G_1} \cap \overline{X}_{G_2}$ for some G_1, G_2. These graphs necessarily differ only by the presence/absence of one edge, which we denote by (i, j).
(b) $(e_i - e_j)^T L_{G_1} x = 0$.
(c) $(e_i - e_j)^T L_{G_1} \neq 0$.

We denote by P the set of problematic points. Observe that $P \subseteq \bigcup_G \partial X_G$. The following lemma characterizes the change of the agent velocities when crossing the boundary between sets.

Lemma 1. *Let $G_{\neg(i,j)}$ be a graph in which i and j are not connected, $G_{(i,j)} = G_{\neg(i,j)} \cup \{(i,j)\}$, and $x \in \partial X_{G_{\neg(i,j)}} \cap \partial X_{G_{(i,j)}}$ such that $x_i > x_j$. We have*

$$-e_i^T L_{G_{\neg(i,j)}} x = -e_i^T L_{G_{(i,j)}} x + 1 \qquad -e_j^T L_{G_{\neg(i,j)}} x = -e_j^T L_{G_{(i,j)}} x - 1.$$

Proof. Since $x_i > x_j$ and $x \in \partial X_{G_{\neg(i,j)}} \cap \partial X_{G_{(i,j)}}$, we have $x_i - x_j = 1$. The result follows because by the definition of L_G,

$$\begin{aligned} -e_i^t L_{G_{(i,j)}} x &= -e_i^T L_{G_{\neg(i,j)}} x + (x_j - x_i) \\ -e_j^t L_{G_{(i,j)}} x &= -e_j^T L_{G_{\neg(i,j)}} x + (x_i - x_j) \end{aligned}$$

□

Lemma 2. *If the procedure described in Section 1 never encounters a problematic point (i.e., if $x(t_k) \notin P$, for all k), it succeeds.*

Proof. Consider step $k+1$ of the procedure, and suppose that $x(t_{k+1})$ is not problematic. We will show that the procedure does then not fail at that step. The first reason for which it could fail is if $x(t_{k+1})$ belongs to the boundary of more than two sets X_G. This would however contradict our assumption that $x(t_{k+1})$ is not problematic. The other possible reason for failure is that there is no $\epsilon > 0$ for which $x^{k+1}(t) \in X_{G_{k+1}}$ for all $t \in (t_{k+1}, t_{k+1}+\epsilon)$, where x^{k+1} is the solution of $\dot{x}^{k+1} = -L_{G_{k+1}} x^{k+1}$ with $x^{k+1}(t_{k+1}) = x(t_{k+1})$.

Since $x(t_{k+1})$ is not problematic, it belongs to the closure of only two sets X_{G_k} and $X_{G_{k+1}}$ and is at a positive distance from all other sets G_k and G_{k+1}, that differ in only one edge (i,j). We assume without loss of generality that $x_i(t_{k+1}) > x_j(t_{k+1})$.

Let us first assume that the edge (i,j) is present in G_k and absent in G_{k+1}. The boundary is $\partial X_{G_{k+1}}$ is locally described by $x_i - x_j = 1$. Any x close to $x(t_{k+1})$ is in X_{G_k+1} if $x_i - x_j > 1$, and in X_{G_k} if $x_i - x_j < 1$. If there is no $\epsilon > 0$ such that $x^{k+1}(t) \in X_{G_{k+1}}$ for all $t \in (t_{k+1}, t_{k+1}+\epsilon)$, the differentiability of x^{k+1} implies that

$$0 \geq \frac{d}{dt}\left(x_i^{k+1}(t_{k+1}) - x_j^{k+1}(t_{k+1})\right) = -(e_i - e_j)^T L_{G_{k+1}} x(t_{k+1}). \tag{3}$$

On the other hand, for $t \in (t_k, t_{k+1})$, $x(t) = x^k(t)$ is by construction in X_{G_k}, and it follows then from the continuity of x and the definition of that set that $x_i^k(t) - x_j^k(t) < 1$ holds on this interval. As a result,

$$-(e_i - e_j)^T L_{G_k} x(t_{k+1}) = \frac{d}{dt}\left(x_i^k(t) - x_j^k(t)\right)\Big|_{t=t_{k+1}} = \lim_{t \uparrow t_{k+1}} \frac{x_i^k(t) - x_j^k(t) - 1}{t - t_k} \geq 0, \tag{4}$$

which by Lemma 1 implies that $-(e_i - e_j)^T L_{G_k} x(t_{k+1}) \geq 2$, contradicting (3).

A contradiction can also be obtained, by a similar argument, in the case where (i,j) is absent in G_k and present in G_{k+1}. Therefore, the procedure succeeds at every step k. □

The proof above only uses the condition that no $x(t_k)$ belongs to the boundary of more than two sets; under this condition, the procedure succeeds and we obtain existence of solutions, even if other problematic points are encountered, satisfying the three conditions (a)-(c). Ruling out this latter kind of problematic points, however, is essential in order to prove uniqueness of solutions, as will become apparent in the proof of the next lemma.

Lemma 3. *If the procedure described in Section 1 never encounters a problematic point (i.e., if $x(t_k) \notin P$, for all k), the function x that it produces is the unique solution of (1) on the domain of definition of x.*

Proof. By Lemma 2, if the procedure never encounters a problematic point, it succeeds and creates a function x that, on its domain of definition, solves (1), and a sequence $(t_0, t_1, \dots)$. Suppose now, to obtain a contradiction, that there exists another function y that solves (1) on the same domain, with $y(0) = x(0)$. Let t^* be the largest time such that $x(t) = y(t)$ holds for all $t \in [0, t^*]$.

Observe first that there cannot exist any $\epsilon > 0$ and graph G such that $x(t), y(t) \in X_G$ for all $t \in (t^*, t^*+\epsilon)$. Indeed, on such an interval, both x and y would be solutions to $\dot{z} = -L_G z$ with $z(t^*) = x(t^*)$, and would thus be identically equal, in contradiction with the definition of t^*. Since x and y are continuous, this implies in particular that $x(t^*)$ must belong to the boundary of (at least) two sets $X_G, X_{G'}$ and therefore that $t^* = t_k$ for some k.

By assumption, $x(t_k)$ is not a problematic point and is thus at positive distance from all sets except $X_{G_{k-1}}$ and X_{G_k}, and G_{k-1} and G_k only differ by the presence or absence of one edge (i,j). Suppose without loss of generality that $x_i(t_k) > x_j(t_k)$, and consider the case where (i,j) is present in G_{k-1} but not in G_k. Note that $(e_i - e_j)^T L_{G_{k-1}} \neq 0$, because otherwise $x_i(t) - x_j(t)$ would be constant on $[t_{k-1}, t_k]$ and x would never reach the boundary of X_{G_k}. Using the argument that led to (3) we must have

$$-(e_i - e_j)^T L_{G_{k-1}} x(t_k) = \lim_{t \uparrow t_k} \frac{d}{dt}\left(x_i(t) - x_j(t)\right) \geq 0. \tag{5}$$

Thus $x(t_k)$ satisfies conditions (a) and (c) for being problematic. Since $x(t_k)$ is not problematic, it must violate condition (b), and we obtain $-(e_i - e_j)^T L_{G_{k-1}} x(t_k) > 0$. Using then the argument that led to (4), we obtain that

$$-(e_i - e_j)^T L_{G_k} x(t_k) > -(e_i - e_j)^T L_{G_{k-1}} x(t_k) > 0.$$

Since y satisfies the integral equation (1), it can be seen that

$$\liminf_{t \downarrow t^*} (e_i - e_j)^T \cdot \frac{y(t) - y(t_k)}{t - t_k} \geq \min \Big\{ -(e_i - e_j)^T L_{G_{k-1}} x(t_k),\ -(e_i - e_j)^T L_{G_k} x(t_k) \Big\} > 0.$$

This implies that there exists some $\epsilon > 0$ such that for $t \in (t_k, t_k + \epsilon)$, both $x(t)$ and $y(t)$ lie in G_k, which we have shown above to be impossible since $t^* = t_k$.

A symmetrical argument applies to the case where (i,j) is absent from G_{k-1} and present in G_k. □

There remains to prove that the unique solution x is defined for all $t > 0$. We have seen that this was the case if the sequence $t_0, t_1, \ldots$ produced by the procedure of Section 1 is finite, and that x was defined on the union of the intervals $[t_k, t_{k+1}]$ if the sequence is infinite. We now show that infinite sequences $(t_0, t_1, \ldots)$ necessarily diverge, and thus that $\bigcup_{k \geq 0} [t_k, t_{k+1}] = \Re^+$.

Lemma 4. *Suppose that the procedure described in Section 1 never encounters a problematic point (i.e., that $x(t_k) \notin P$, for all k), and produces an infinite sequence of transition times $t_0, t_1, \ldots$. Then, this sequence diverges, and therefore there exists a unique solution x, defined for all $t \geq 0$.*

Proof. Since the sequence $t_1, t_2, \ldots$ of transition times is infinite, a nonempty set of agents is involved in an infinite number of transitions, and there exists a time T after which every agent involved in a transition will also be involved in a subsequent one. Consider now a transition occurring at $s_1 > T$ and involving agents i and j. We denote by $\dot{x}_i(s_1^-)$ and $\dot{x}_i(s_1^+)$ the limits $\lim_{t \uparrow s_1} \dot{x}_i(t)$ and $\lim_{t \downarrow s_1} \dot{x}_i(t)$ respectively. (Note that these limits exist because away from boundary points, the function x is continuously differentiable.)

Suppose without loss of generality that $x_i > x_j$. We consider two cases. (i) Suppose that i and j are connected before time s_1 but not after. The update equation (1) implies that $\dot{x}_i(s_1^+) = \dot{x}_i(s_1^-) - (x_j(s_1) - x_i(s_1))$. Noting that $x_i(s_1) - x_j(s_1) = 1$, we conclude that that $\dot{x}_i(s_1^+) = \dot{x}_i(s_1^-) + 1$. Moreover, $x_i - x_j$ must have been increasing just before s_1, so that $\dot{x}_i(s_1^-) \geq \dot{x}_j(s_1^-)$. (ii) Suppose now that i and j are connected after s_1 but not before. Then, $\dot{x}_j(s_1^+) = \dot{x}_j(s_1^-) + 1$, and since $x_i - x_j$ must have been decreasing just before s_1, we must have $\dot{x}_j(s_1^-) \geq \dot{x}_i(s_1^-)$. In either case, there exists an agent $k_1 \in \{i, j\}$ for which $\dot{x}_{k_1}(s_1^+) = \max\{\dot{x}_i(s_1^-), \dot{x}_j(s_1^-)\} + 1$. It follows from $s_1 > T$ that this agent will get involved in some other transition at a further time. Call s_2 the first such time.

The definition (1) of the system implies that in between transitions, $|\dot{x}_i(t)| \leq n$ for all agents. Using (1) again, this implies that $|\ddot{x}_i(t)| \leq 2n^2$ for all t at which i is not involved in a transition. Therefore, $\dot{x}_{k_1}(s_2^-) \geq \dot{x}_{k_1}(s_1^+) - 2n^2(s_2 - s_1) = x_i(s_1^-) + 1 - 2n^2(s_2 - s_1)$. Moreover, by the same argument as above, there exists a k_2 for which $\dot{x}_{k_2}(s_2^+) = \dot{x}_{k_1}(s_2^-) + 1 \geq x_i(s_1^-) + 2 - 2n^2(s_2 - s_1)$. Continuing recursively, we can build an infinite sequence of transition times $s_1, s_2, \ldots$ (a subsequence of $t_1, t_2, \ldots$), such that for every m,

$$\dot{x}_{k_m}(s_m^+) \geq \dot{x}_i(s_1^-) + m - 2n^2(s_m - s_1).$$

holds for some agent k_m. Since all velocities are bounded by n, this implies that $s_m - s_1$ must diverge as m grows, and therefore that the sequence $t_1, t_2, \ldots$ of transition times diverges. □

The proof of Lemma 4 also provides an explicit bound on the number of transitions that can take place during any given time interval. The following proposition summarizes the results of this section.

Proposition 1. *If $\tilde{x} \in \Re^n \setminus (\bigcup_G \partial X_G)$ is not a proper initial condition, then the procedure of Section 1 will reach some problematic point $x(t_k) \in P \subseteq \bigcup_G \partial X_G$.*

Proof. It follows from Lemmas 2, 3 and 4 that if $\tilde{x}$ does not satisfy condition (a) or (b) for being proper, the procedure of Section 1 fails at some problematic point. Observe now that any $\tilde{x}$ satisfying condition (a) for being proper also satisfies condition (c). To see this, suppose that x is a solution of (1) with $x(0) = \tilde{x}$, and that $x_i(t) = x_j(t)$ holds for some t, but not for some latter $t' > t$. One can then build another solution by switching x_i and x_j after time t. It follows that x is not a unique solution of (1), and $\tilde{x}$ does not satisfy condition (a) for being proper. □

Note that the absence of a problematic transition point is sufficient but not a necessary condition for $\tilde{x}$ to be proper. In particular, the initial condition $\tilde{x} = \frac{1}{2}(-1,-1,0,1,1)$ is a problematic point because it belongs to the closure of more than two sets. There holds indeed $|\tilde{x}_1 - \tilde{x}_4| = |\tilde{x}_1 - \tilde{x}_5| = |\tilde{x}_2 - \tilde{x}_4| = |\tilde{x}_2 - \tilde{x}_5| = 1$. However, one can verify that Eq. (1) has a unique and proper solution $x(t) = \tilde{x}e^{-5t}$.

3 Measure of the set of non-proper initial conditions

We now show that unless $\tilde{x}$ belongs to a certain zero measure set, the procedure starting with $\tilde{x}$ never encounters a problematic point. We denote by μ the natural measure on the $n-1$ dimensional set $\bigcup_G \partial X_G$, and begin by proving that $\mu(P) = 0$. The proof consists of showing that P is the union of $(n-2)$-dimensional affine spaces.

Lemma 5. *The set P of problematic points measure has zero measure in $\bigcup_G \partial X_G$.*

Proof. Observe first that the points that belong to the boundary of more than two sets necessarily satisfy $x_i - x_j = \pm 1$ and $x_p - x_q = \pm 1$, with at least one of p or q different from i and j. The set of these points is thus included in a finite union of $(n-2)$-dimensional affine spaces, and has zero measure in the $(n-1)$-dimensional set $\bigcup_G \partial X_G$.

Consider now a point x satisfying the three conditions (a)-(c) for being problematic. Condition (a) implies that $x_i - x_j = \pm 1$ for some i, j. Moreover, it follows from condition (b) that $(e_i - e_j)^T L_{G_1} x = 0$ holds for some graph G_1. Condition (c) implies that the set of points satisfying the last equality is a $(n-1)$-dimensional space. The latter set is clearly not identical to the $(n-1)$-dimensional affine space defined by $x_i - x_j = \pm 1$, so that their intersection, if it exists, is an affine space of dimension $n-2$. Therefore, the set of points x satisfying the three conditions (a)-(c) is included in a finite union of $(n-2)$-dimensional sets, and thus has zero measure in $\bigcup_G \partial X_G$. □

Let $P_0 = P$, and for every $k > 0$, let P_k be the set of points in $y \in \bigcup_G \partial X_G$ for which there exists G and $t^* > 0$ such that:
a) $y \in \partial X_G$,
b) $e^{-L_G t} y \in X_G$, for all $t \in (0, t^*)$,
c) $e^{-L_G t^*} y \in P_{k-1}$.
Thus, P_k is the set of points, on some boundary, from which we can reach a problematic point after k transitions. Points in P_k, as well as their pre-images are the only ones that can eventually lead to a problematic point and therefore destroy properness. This is stated in the next lemma, which does not require further proof. Note that $e^{L_G t}$ is the transition matrix under the inverse dynamics.

Lemma 6. *If $\tilde{x} \in \Re^n$ is not a proper initial condition, then either $\tilde{x} \in \bigcup_G \partial X_G$ or*

$$\tilde{x} \in \bigcup_G \left\{ e^{L_G t} y : t \in \Re^+, y \in \bigcup_k P_k \right\} \tag{6}$$

In order to show that the set in (6) also has a zero measure, and complete the proof of Theorem 1, we will use Proposition 2 below, which is proved in Section 1.

Let H_0, H_1 be two affine subspaces in $\Re^n$, and let A be a $n \times n$ matrix. We define a partial function $g : H_0 \to H_1$ as follows. If there exists a time $t > 0$ such that:
(i) $y = e^{At} x \in H_1$,
(ii) $e^{At'} x \notin H_1$ for all $t' \in (0, t)$,
we let $g(x) = y$. Otherwise, $g(x)$ is left undefined. In words, if starting from $x \in H_0$, the solution of the differential equation $\dot{x} = Ax$ eventually hits H_1, then $g(y)$ is the first point in H_1 that lies on the trajectory.

Proposition 2. *Suppose that H_0 and H_1 have dimension $n-1$, and that H_1 is not a subspace (i.e., does not contain the zero vector). If $X \subseteq H_0$ has zero measure (in H_0), then $g(X)$ has zero measure (in H_1).*

Lemma 7. *We have $\mu(\bigcup_k P_k) = 0$, and the set*

$$\bigcup_G \left\{ e^{L_G t} y : t \in \Re^+, y \in \bigcup_k P_k \right\}$$

has thus zero measure in $\Re^n$.

Proof. We have proved in Lemma 5 that $\mu(P_0) = 0$. We now assume that $\mu(P_k) = 0$; we will prove that $\mu(P_{k+1}) = 0$. Observe first that

$$P_k, P_{k+1} \subseteq \bigcup_G \partial X_G = \bigcup_{i,j} \{x \in \Re^n : x_i - x_j = 1\}.$$

Consider two (possibly identical) hyperplanes $H_{i,j}, H_{p,q}$ defined by $x_i - x_j = 1$ and $x_p - x_q = 1$.

Fix a graph G and let $g_{i,j,p,q,G}$ be a function defined on the largest possible subset of $H_{i,j}$, which maps x to a particular $y \in H_{p,q}$ if there exists a t_x such that $y = e^{L_G t_x} x \in H_{p,q}$ and $e^{L_G t} x \notin H_{p,q}$ for all $t \in (0, t_x)$.

By the definition of $g_{i,j,p,q,G}$ and P_{k+1}, if $y \in H_{p,q} \cup P_{k+1}$, then there exists a graph G, indices i, j, and $x \in H_{i,j}$ such that $y = g_{i,j,p,q,G}(x)$. Therefore,

$$P_{k+1} \subseteq \bigcup_{i,j,p,q,G} H_{p,q} \cap g_{i,j,p,q,G}(P_k \cap H_{i,j}).$$

Since $\mu(P_k) = 0$, Proposition 2 implies that $\mu(P_{k+1}) = 0$. The induction is complete and shows that $\mu(P_k) = 0$ for all k. Therefore, the countable union of all P_k also has zero measure in the $n-1$ dimensional set $\bigcup_G \partial X_G$. It follows that

$$\bigcup_G \left\{ e^{L_G t} y : t \in \Re^+, y \in \bigcup_k P_k \right\}$$

has zero measure in $\Re^n$. □

Together with Lemma 6, this last result completes the proof of Theorem 1.

4 Proof of Proposition 2

The proof will make use of the following two facts:

(F1) Let B be an open subset. Suppose that the function $f : B \to \Re^n$ is infinitely differentiable and has nonsingular Jacobian at every $x \in B$. If $S \subseteq B$ has positive Lebesgue measure, then so does $f(S)$.

(F2) Let B be an open subset. Suppose that the function $f : B \to \Re^n$ is infinitely differentiable. If $S \subseteq B$ has zero Lebesgue measure, then so does $f(S)$.

The first fact is true because the mapping f is locally a diffeomorphism, and the Jacobian formula for the transformation of measures applies. The second fact can be found, for example in [2] (Lemma 10.2, p. 243).

The $(n-1)$-dimensional affine subspace H_1 is necessarily of the form $\{x : p'x = c\}$, for some constant $c \neq 0$, and some unit length vector p. Suppose first that $p'Ax = 0$ for every $x \in H_1$. Since H_1 is an affine subspace, of dimension $n-1$, but is not a subspace (does not contain the origin), H_1 contains n linearly independent vectors. It follows that $p'A = 0$. This implies that $p'x$ is constant along any trajectory of the differential equation $\dot{x} = Ax$. In particular, any trajectory that starts outside H_1 will never reach H_1. In this case, the map g is undefined, for every $x \in H_0$. We can therefore assume from now on that there exists some $x \in H_1$ for which $p'Ax \neq 0$.

Consider now the set $[0, \infty) \times H_0$, which can be identified with a halfspace in $\Re^n$, and endowed with (n-dimensional) Lebesgue measure. Consider the C_∞ mapping h defined by

$$h : [0, \infty) \times H_0 \to \Re^n : \ (t, x) \to e^{At} x.$$

The image $h(X)$ of that mapping, to be denoted by Q, is the set of all points in $\Re^n$ reachable from X. Since X has zero measure in H_0, it follows (from Fubini's theorem) that $[0,\infty) \times X$ has zero measure. Using Fact (F2), it follows that Q, has zero (n-dimensional) Lebesgue measure.

Suppose now, in order to derive a contradiction, that $g(X)$ has positive ($(n-1)$-dimensional) measure in H_1. Consider the subspace on which $p'Ax = 0$. Since $p'A \neq 0$, this is a proper subspace. The intersection of this subspace with the affine space H_1 (which does not contain the origin) is a set of dimension at most $n-2$. Therefore, this intersection has zero measure in H_1. After removing this zero measure set, we are left with a positive measure subset of $g(X)$ on which $p'Ax \neq 0$. Without loss of generality, assume that there is a positive measure subset of $g(X)$ on which $p'Ax$ is positive. It follows that we can find some $\delta > 0$, such that the set

$$g(X) \cap \{x : p'Ax > 2\delta\}$$

has positive measure (in H_1). By a routine argument, and using the continuity of $p'Ax$ as a function of x, we can find an open cube in $\Re^n$, to be denoted by B, on which $p'Ax > \delta$ and whose intersection with $g(X)$ has positive measure (in H_1).

Fix a small constant $\eta > 0$. Consider the mapping $s : (-\eta, \eta) \times (B \cap H_1) \to \Re^n$, that maps (t, x) to $e^{At}x$. Because of the transversality condition $p'Ax > \delta$, and by taking η sufficiently small, it is not hard to show that the Jacobian of this mapping is nonsingular. It follows from Fact (F1), that the mapping s maps positive measure sets to positive measure sets. Consider the set $[0, \eta) \times g(X)$; since $g(X) \cap B$ has positive measure (in H_1), it follows that $[0, \eta) \times g(X)$ has positive measure. Therefore, the set

$$D = s\big([0, \eta) \times g(X)\big) = \{e^{At}x : \ t \in [0, \eta), \ x \in g(X)\},$$

has positive measure (in $\Re^n$). Recall now that every point in $g(X)$ is reachable from X. This implies that points in D are also reachable from X. But then, as shown earlier, the measure of D must be zero. This is a contradiction and concludes the proof of the proposition.

Acknowledgement

The authors are pleased to acknowledge discussions with Prof. Eduardo Sontag on the assertion and proof of Proposition 2.

References

[1] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis. Continuous-time average-preserving opinion dynamics with opinion-dependent communications. *Preprint*, 2009.

[2] J.M. Lee. *Introduction to Smooth Manifolds.* Springer Verlag, 2009.