The Bond-Algebraic Approach to Dualities

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Bond algebra is an approach to duality in statistical mechanics that focuses on the local interaction terms (bonds) in a Hamiltonian or a transfer matrix, which can form a von Neumann algebra. This algebra encapsulates the structural and symmetry properties of the system, providing a general framework to capture universal features of phase transitions and critical phenomena. In this paper, I review the basic definition of this approach and its applications to dualities in both the quantum and classical models. I will also use it to derive a duality between models that are modified versions of the $D = 3$ classical vector Potts and $\mathbb{Z}_p$ gauge models. This is a $p-$state generalization of the well known duality between the $D = 3$ Ising model and the $\mathbb{Z}_2$ gauge model.

I. INTRODUCTION

Duality connects two different realizations of essentially the same object and is pervasive in physics. The oldest well-known example is probably the electromagnetic duality of Maxwell's equations in the absence of sources. The duality has also been popular in field theories such as Anti de Sitter–conformal field theory (AdS–CFT) correspondence. In the context of statistical physics, Kramers and Wannier[1] first introduced the duality for the two-dimensional Ising model on a square lattice between the high and low temperature phases, and used it to determine the exact critical temperature. This duality focused on the partition functions of classical statistical models and was referred to classical duality in this paper. Later, it was generalized to models with general elementary degrees of freedom and lattices with the aid of the Fourier transformation[2, 3]. Another concept of duality was developed at the same time for quantum many-body problems concerned with connections between two quantum Hamiltonians[4], which is referred to as quantum duality here. It established a unitary equivalence between two Hamiltonians by finding two representations of the algebra of the elementary degrees of freedom (e.g., two sets of spin operators on lattice sites satisfying the same commutation relations). In a broader sense, the quantum duality can include duality mappings that change the elementary statistics. For instance, spin 1/2 operators are mapped onto spinless fermions by a transformation known as the Jordan–Wigner transformation[5], which can also be regarded as a duality. Such duality transformations are valuable because they provide exact solvability in one-dimensional systems.

In 2010, Cobanera *et al.* introduced a new approach to both classical and quantum dualities in a unified way[6, 7]. In contrast to the traditional approach, their theory of dualities rests on the observation that the bonds or interactions of a Hamiltonian or a transfer matrix are of more relevance to a duality transformation than the elementary degrees of freedom. Those bonds, or interactions, are organized into a bond algebra, which is a Von Neumann algebra. The dualities are then structure-preserving mappings (homomorphisms) of bond algebras, typically local in the bonds. The crucial difference with the traditional approach is that the bond algebras are model-specific. By choosing different decompositions of the bonds, one can achieve different dualities for the same model. On the other hand, different models may share the same bond algebra, so one model's duality can be transferred directly to another model's duality. This approach can be systematically applied to more general models and is especially efficient for lattice gauge theory, where the bond algebra can be utilized to eliminate gauge constraints.

In this paper, I will summarize the bond algebra approach to duality and use it to find some dualities of statistical models. Section. II starts with the basic definition and properties of the bond algebra. The rest of the paper is devoted to this method to establish dualities. Section. III investigate the quantum duality with the example of lattice $\mathbb{Z}_p$ gauge theory. It also illustrates the idea of gauge-reducing duality by mapping the $\mathbb{Z}_p$ model to the vector Potts (VP) model without any local gauge symmetry. Section. IV presents the dualities of classical statistical models. In particular, I show its connection with quantum duality and how the bond algebra approach can transform a quantum duality into a classical duality.

II. BOND ALGEBRA BASICS

A. Definition

To define the bond algebra, consider a Hamiltonian operator H, written as a sum of bond operators h_Γ

$$H = \sum_\Gamma \lambda_\Gamma h_\Gamma \tag{1}$$

with c-number coupling constants λ_Γ. The index " Γ " is completely general. It could stand for a particle index, or

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a site, a link, or some other subregions of a lattice Λ. A bond algebra for the Hamiltonian H is defined as the von Neumann algebra $A\{h_\Gamma\}$ generated by the bonds $\{h_\Gamma\}$. To be specific, it's the sub-algebra of $B(\mathcal{H})$ (the algebra of bounded operators on Hilbert space $\mathcal{H}$), generated by the bonds, that satisfies the following conditions: (1) It contains the identity operator, $I \in \mathcal{A}$. (2) It is closed under Hermitian conjugation, if $\mathcal{O} \in \mathcal{A}$, then $\mathcal{O}^\dagger \in \mathcal{A}$ as well. (3) It is equal to its bycommutant, $\mathcal{A} = \mathcal{A}''$. The commutant of a subset $\mathcal{S}$ is defined by

$$\mathcal{S}' = \{\mathcal{O} \in B(\mathcal{H}) \mid \forall \mathcal{R} \in \mathcal{S}, \mathcal{O}\mathcal{R} = \mathcal{R}\mathcal{O}\}. \tag{2}$$

Intuitively speaking, $\mathcal{A}\{h_\Gamma\}$ is an algebra of operators generated by taking all possible finite, complex, linear combinations of powers and products of bonds, their Hermitian conjugates, and the identity operator I,

$$\left\{\not\Vdash, h_\Gamma, h_\Gamma^\dagger, h_\Gamma h_{\Gamma'}, h_{\Gamma'}^\dagger h_{\Gamma'}, h_{\Gamma'}^\dagger h_\Gamma, h_{\Gamma'}^\dagger h_{\Gamma'}^\dagger, h_\Gamma h_{\Gamma'} h_{\Gamma''}, \ldots\right\}. \tag{3}$$

A mapping of von Neumann algebras $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an homomorphism if

$$\begin{aligned} \Phi(\mathbb{I}) = \mathbb{I}, \quad \Phi\left(\mathcal{O}^\dagger\right) &= \Phi(\mathcal{O})^\dagger \\ \Phi\left(\mathcal{O}_1\mathcal{O}_2\right) &= \Phi\left(\mathcal{O}_1\right)\Phi\left(\mathcal{O}_2\right), \\ \Phi\left(\mathcal{O}_1 + \lambda\mathcal{O}_2\right) &= \Phi\left(\mathcal{O}_1\right) + \lambda\Phi\left(\mathcal{O}_2\right). \end{aligned} \tag{4}$$

If Φ is one-to-one and onto, it's called an isomorphism. A theorem[8] guarantees that Φ can be implemented as an isometry: $\Phi(\mathcal{O}) = \mathcal{U}\mathcal{O}\mathcal{U}^\dagger$ and $\mathcal{U}^\dagger\mathcal{U} = I$. For models without gauge symmetry, the bond algebra approach to dualities is that *quantum dualities are isomorphisms of bond algebras*. Hence, the duality mapping Φ is an isomorphism and $\mathcal{U}$ is a unitary. The gauge-reducing dualities and classical dualities will be discussed later.

B. Quantum Ising chain

To illustrate these concepts, consider the quantum Ising chain (one-dimensional transverse field Ising model)

$$H_{\mathrm{I}}[h, J](\sigma) = \sum_i \left(h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z\right). \tag{5}$$

H_{I} features $S = \frac{1}{2}$ spins located at each site $i \in \mathbb{Z}$ of a chain, represented by Pauli matrices σ_i^x and σ_i^z. Take the basic bonds in H_I of Equation (15) to be $\left\{\sigma_i^z\sigma_{i+1}^z\right\}, \{\sigma_i^x\}$. They generate a bond algebra $\mathcal{A}_{\mathrm{I}}$ that is characterized by: (1) $\left(\sigma_i^z\sigma_{i+1}^z\right)^2 = I = (\sigma_i^x)^2$; (2) Any bond σ_i^x anti-commutes with two other bonds, $\sigma_{i-1}^z\sigma_i^z$ and $\sigma_i^z\sigma_{i+1}^z$, and commutes with all other bonds; (3) Any bond $\sigma_i^z\sigma_{i+1}^z$ anti-commutes with two other bonds, σ_i^x and σ_{i+1}^x, and commutes with all other bonds. It's easy to see that σ_i^x and $\sigma_i^z\sigma_{i+1}^z$ play perfectly symmetrical roles, and so we can set up the mapping

$$\Phi_{\mathrm{d}}\left(\sigma_i^z\sigma_{i+1}^z\right) = \sigma_i^x, \qquad \Phi_{\mathrm{d}}\left(\sigma_i^x\right) = \sigma_{i-1}^z\sigma_i^z. \tag{6}$$

and extends it to a unique isomorphism of the full bond algebra $\mathcal{A}_{\mathrm{I}}$. It follows that the quantum Ising chain is self-dual

$$H_{\mathrm{I}}[h, J] \xrightarrow{\Phi_{\mathrm{d}}} H_{\mathrm{I}}[J, h]. \tag{7}$$

Therefore, its energy levels are symmetric in J and h ($EI(J, h) = EI(h, J)$) and it displays a quantum phase transition at the self-dual point $h = J$.

As pointed out before, this duality Φ_{d} can be applied to Hamiltonians other than HI, as long as they are affiliated to $\mathcal{A}_{\mathrm{I}}$ (whose bond algebra is a sub-algebra of $\mathcal{A}_{\mathrm{I}}$). can apply Φ_{d} to Hamiltonians other than H_{I}, as long as they are affiliated to $\mathcal{A}_{\mathrm{I}}$. For example, consider the one-dimensional spin $S = \frac{1}{2}$ XY-model,

$$H_{\mathrm{XY}} = \sum_i \left(J_x\sigma_i^x\sigma_{i+1}^x + J_z\sigma_i^z\sigma_{i+1}^z\right). \tag{8}$$

The bonds $\sigma_i^z\sigma_{i+1}^z$ of H_{XY} are already bonds of H_{I}. The $\sigma_i^x\sigma_{i+1}^x$ in H_{XY} are the products of two bonds of H_{I}. Thus, it is possible to use the isomorphism of the quantum Ising model to compute a dual form of the XY-model. As $\sigma_i^x\sigma_{i+1}^x \xrightarrow{\Phi_d} \sigma_{i-1}^z\sigma_i^z\sigma_i^z\sigma_{i+1}^z$, we find that

$$H_{\mathrm{XY}} \xrightarrow{\Phi_{\mathrm{d}}} H_{\mathrm{Inn}} = \sum_i \left(J_x\sigma_{i-1}^z\sigma_{i+1}^z + J_z\sigma_i^x\right). \tag{9}$$

H_{Inn} is trivially dual to two decoupled Ising chains. The fact that, in $d = 1$, H_{XY} and H_{Inn} share the same energy spectra was first noted by Pfeuty[9].

C. Determination of dual variables

The traditional approach to dualities focuses on dual variables that are the operator change of non-local variables. To connect with it, the isomorphism of bond algebras can be used to uniquely determine the problem's dual variables. However, duality mappings established in the limit of infinite size are well defined only on finite combinations of bonds, but have ill-defined actions on infinite combinations of bonds[7]. The naive extension of the action of duality mappings by homomorphism will cause multi-valued problems. The practical solution is to work with bond algebras of finite-size systems with suitably chosen boundary conditions to restore the exact duality. Take the quantum Ising model for example, let's consider its finite-size rendition

$$H_{\mathrm{I}}^*[h, J] = -J\sum_{j=1}^{N-1}\sigma_j^z\sigma_{j+1}^z - h\sum_{j=1}^{N-1}\sigma_j^x. \tag{10}$$

Notice that the transverse field h is not applied to site N. This special arrangement makes it possible to formulate an exact quantum duality.

The exact self-duality isomorphism is constructed as

$$\begin{aligned}
&\sigma_i^x \xrightarrow{\Phi_{\rm d}} \sigma_{r(i)-1}^z\sigma_{r(i)}^z, \quad i=1,2,\ldots,N-1,\\
&\sigma_i^z\sigma_{i+1}^z \xrightarrow{\Phi_{\rm d}} \sigma_{r(i)-1}^x, \quad i=1,2,\ldots,N-1,\\
&\sigma_N^x \xrightarrow{\Phi_{\rm d}} \sigma_1^z,\\
&\sigma_1^z \xrightarrow{\Phi_{\rm d}} \sigma_N^x.
\end{aligned} \tag{11}$$

$r(i) = N+1-i$ represents the inversion map. It's easy to check the first two lines of Eq.(11) preserve the anti-commutation relations and give the self-duality: $\Phi_{\rm d}(H_{\rm I}^*[h,J]) = H_{\rm I}^*[J,h]$. The bonds σ_N^x and σ_1^z are also added to the bond algebra $\mathcal{A}_{\rm I}^*$ and the isomorphism $\Phi_{\rm d}$ is extended to act on them as in the last two lines of Eq.(11). Those extra bonds guarantee that the individual spins $\sigma_i^z, i=1,\ldots,N$ are elements in the bond algebra, since we can write

$$\sigma_i^z = \sigma_i^z\sigma_{i-1}^z \times \sigma_{i-1}^z\sigma_{i-2}^z \times \cdots \times \sigma_2^z\sigma_1^z \times \sigma_1^z. \tag{12}$$

Then the dual variables $\mu_i^{x,y,z} \coloneqq \Phi_{\rm d}\left(\sigma_i^{x,y,z}\right)$ are

$$\begin{aligned}
&\mu_i^x = \sigma_{r(i)-1}^z\sigma_{r(i)}^z, \quad i=1,2,\ldots,N-1,\\
&\mu_N^x = \sigma_1^z,\\
&\mu_i^z = \prod_{m=i}^{N}\sigma_{r(m)}^x.
\end{aligned} \tag{13}$$

This gives essentially the same results as the traditional approach focusing on the elementary spin operators[4]. Notice that even the dual bonds are strictly local, the dual variables (μ_i^z) are usually non-local in space, which are represented as products of a string of spin operators.

With the finite rendition, it's also straightforward to explicitly construct the unitary operator $\mathcal{U}_{\rm d}$ that implements the isomorphism $\Phi_{\rm d}$. One can refer to [7] for such an expression.

D. Duality and quantum symmetry

The connection between dualities and symmetries is most explicit for self-dual models. As argued, self-dualities are usually unitarily implementable transformations and preserve the form of the Hamiltonian $\mathcal{H}$. However, self-duality will change the coupling parameters, so it's not a symmetry of $\mathcal{H}$. Nevertheless, there is a close relation between self-duality and symmetry. Suppose, for simplicity, that we have a Hamiltonian $\mathcal{H}[\lambda_1,\lambda_2,\cdots]$, dependent upon a set of couplings λ_ν, that is self-dual under the exchange $\lambda_1 \leftrightarrow \lambda_2$, that is

$$\mathcal{U}_{\rm d} H\left[\lambda_1,\lambda_2,\ldots\right]\mathcal{U}_{\rm d}^\dagger = H\left[\lambda_2,\lambda_1,\ldots\right], \tag{14}$$

with $\mathcal{U}_{\rm d}$ a unitary independent of the couplings. $\mathcal{U}_{\rm d}$ relates to symmetries of $\mathcal{H}$ in two ways. First,

$$\left[H\left[\lambda_1,\lambda_2,\ldots\right],\mathcal{U}_{\rm d}^{2n}\right] = 0, \tag{15}$$

i.e. $\mathcal{U}_d^{2n}$ are symmetries of H for any $n=1,2\ldots$. Second, Eq.(14) shows that at the self-dual point $\lambda_1=\lambda_2$, $\mathcal{U}_{\rm d}$ commutes with $\mathcal{H}$. Hence, $\mathcal{U}_{\rm d}$ emerges as a new symmetry at the self-dual point.

Again, consider the isomorphism (6) of the quantum Ising model (5). Due to the duality of (7), $\mathcal{U}_d^2$ commutes with $H_{\rm I}$. Indeed, it's just the translational symmetry

$$\mathcal{U}_{\rm d}^2\sigma_i^x\mathcal{U}_{\rm d}^{\dagger 2} = \sigma_{i-1}^x, \quad \mathcal{U}_{\rm d}^2\sigma_i^z\sigma_{i+1}^z\mathcal{U}_{\rm d}^{\dagger 2} = \sigma_{i-1}^z\sigma_i^z. \tag{16}$$

III. QUANTUM DUALITY

A. Gauging-reducing dualities

Lattice gauge theories are models with local symmetries that look like gauge symmetries. In general, gauge symmetries are constraints pointing to a fundamental redundancy. The state space of a model with gauge symmetries is larger than physical, which contains states that cannot be prepared or observed by experimental means. The sector of physical states is precisely that sector that is invariant under the action of all the local gauge symmetries. It turns out that bond-algebraic dualities are practical tools for removing gauge symmetries. If one chooses the bond algebra of the gauge model wisely, one can find mappings that preserve all the algebraic relationships to models that do not have any gauge symmetries.

To be specific, let H_G be the Hamiltonian for the gauge model, with gauge symmetries G, $[H_G, G]=0$, and let H_{GR} be the dual, completely gauge-reduced model. Then the gauge-reducing duality maps

$$\Phi_{\rm d}\left(H_{\rm G}\right) = H_{\rm GR} \qquad \Phi_{\rm d}\left(G_\Gamma\right) = I \quad \forall\Gamma, \tag{17}$$

thus rendering all the gauge symmetries trivial. Note $\Phi_{\rm d}$ will change the number of degrees of freedom and cannot be implemented unitarily. Indeed, $\Phi_{\rm d}$ is implemented as an *isometry* (rectangular matrices) $\mathcal{U}_{\rm d}^2$ that preserves the norm of gauge-invariant states, and projects other states out.

$$\Phi_{\rm d}(\mathcal{O}) = U_{\rm d}\mathcal{O}U_{\rm d}^\dagger \tag{18}$$

with

$$U_{\rm d}U_{\rm d}^\dagger = I, \quad U_{\rm d}^\dagger U_{\rm d} = P_{\rm GI}, \tag{19}$$

with $P_{\rm GI} = P_{\rm GI}^2 = P_{\rm GI}^\dagger$ is the orthogonal projector onto the subspace of gauge invariant states that satisfy

$$G_\Gamma|\psi\rangle = |\psi\rangle \quad \forall\Gamma. \tag{20}$$

B. Two-dimensional $\mathbb{Z}_p$ gauge/vector Potts models

As an example of the gauge-reducing dualities, let's consider the $d=2$ dimensional $\mathbb{Z}_p$ gauge theory,

$$H_{\rm G} = -\frac{1}{2}\sum_{\boldsymbol{r}}\left(V_{(\boldsymbol{r},1)} + V_{(\boldsymbol{r},2)} + \lambda B_{(\boldsymbol{r},3)} + \text{h.c.}\right). \tag{21}$$

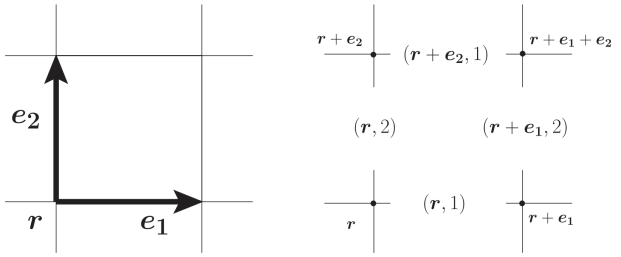

FIG. 1. (Left panel) Convention to denote vertices $\boldsymbol{r} = (r^1, r^2) = r^1\boldsymbol{e}_1 + r^2\boldsymbol{e}_2$ in a two-dimensional square lattice with unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, and (right panel) links, attached to a vertex $\boldsymbol{r}$, $(\boldsymbol{r}, v)$ with $\nu = 1, 2$.

with

$$B_{(\boldsymbol{r},3)} \coloneqq U_{(\boldsymbol{r},1)}U_{(\boldsymbol{r}+\boldsymbol{e}_1,2)}U^\dagger_{(\boldsymbol{r}+\boldsymbol{e}_2,1)}U^\dagger_{(\boldsymbol{r},2)}. \tag{22}$$

The operators $U(\boldsymbol{r},\nu)$ and $V(\boldsymbol{r},\nu)$, where $\nu = 1, 2$, located at the links (see Figure 2) of a square lattice (see Figure.1), commute on different links, and satisfy the Weyl group algebra (Appendix.A). The (21) is a generalization of the Ising gauge model and reduces to it when $p = 2$. $H_{\rm G}$ displays a $\mathbb{Z}_p$ gauge symmetry as realized by the local symmetry operators

$$G_{\boldsymbol{r}} = V_{(\boldsymbol{r},1)}V_{(\boldsymbol{r},2)}V^\dagger_{(\boldsymbol{r}-\boldsymbol{e}_1,1)}V^\dagger_{(\boldsymbol{r}-\boldsymbol{e}_2,2)}. \tag{23}$$

We want to find a transformation to a dual Hamiltonian that is free of gauge symmetries. The $d = 2$ dimensional quantum VP model

$$H_{\rm VP} = -\frac{1}{2}\sum_{\boldsymbol{r}}\left(\lambda V_{\boldsymbol{r}} + U_{\boldsymbol{r}}U^\dagger_{\boldsymbol{r}+\boldsymbol{e}_1} + U_{\boldsymbol{r}}U^\dagger_{\boldsymbol{r}+\boldsymbol{e}_2} + \text{ h.c. }\right). \tag{24}$$

turns out to be the desired model. $H_{\rm VP}$ is a generalization of the $d = 2$ transverse field Ising model in terms of Weyl group algebra operators. The bond algebra homomorphism is

$$B_{(\boldsymbol{r},3)} \xrightarrow{\Phi_d} V_{\boldsymbol{r}}, \quad V_{(\boldsymbol{r},1)} \xrightarrow{\Phi_d} U_{\boldsymbol{r}-\boldsymbol{e}_2}U^\dagger_{\boldsymbol{r}}, \quad V_{(\boldsymbol{r},2)} \xrightarrow{\Phi_d} U^\dagger_{\boldsymbol{r}-\boldsymbol{e}_1}U_{\boldsymbol{r}}, \tag{25}$$

which completes the elimination of gauge symmetries,

$$\begin{aligned}\Phi_{\rm d}\left(G_{\boldsymbol{r}}\right) &= U_{\boldsymbol{r}-e_2}U^\dagger_{\boldsymbol{r}} \times U^\dagger_{\boldsymbol{r}-\boldsymbol{e}_1}U_{\boldsymbol{r}} \\ &\quad \times U^\dagger_{\boldsymbol{r}-\boldsymbol{e}_2-\boldsymbol{e}_1}U_{\boldsymbol{r}-\boldsymbol{e}_1} \times U_{\boldsymbol{r}-\boldsymbol{e}_1-\boldsymbol{e}_2}U^\dagger_{\boldsymbol{r}-\boldsymbol{e}_2} \\ &= I.\end{aligned} \tag{26}$$

This homomorphism affords a simple and conceptually clarifying proof that $H_{\rm VP}$ encodes the observable, gauge-invariant physics of $H_{\rm G}$.

For later convenience, here I propose a duality between the generalized $H_{\rm G}$ and $H_{\rm VP}$. The key is to include all the powers of Weyl generators in $H_{\rm G}$ and $H_{\rm VP}$,

$$\begin{aligned}H^*_{\rm G}[\mu,\lambda] &= -\sum_{m=0}^{p-1}\sum_{\boldsymbol{r}}\left(\mu_m\sum_{\nu=1,2}V^m_{(\boldsymbol{r},\nu)} + \lambda_m B^m_{(\boldsymbol{r},3)}\right) \\ &= H^0_{\rm G}[\lambda] + H^1_{\rm G}[\mu].\end{aligned} \tag{27}$$

and

$$\begin{aligned}H^*_{\rm VP}[\mu,\lambda] &= -\sum_{m=0}^{p-1}\sum_{\boldsymbol{r}}\left(\lambda_m V^m_{\boldsymbol{r}} + \mu_m\sum_{\nu=1,2}(U_{\boldsymbol{r}}U^\dagger_{\boldsymbol{r}+\boldsymbol{e}_\nu})^m\right) \\ &= H^0_{\rm VP}[\mu] + H^1_{\rm VP}[\lambda]\end{aligned} \tag{28}$$

The coefficients λ_m, μ_m are real and symmetric ($\lambda_m = \lambda_{p-m}$, same for μ_m). Since $H^*_{\rm G}$ and $H^*_{\rm VP}$ are affiliated to the bond algebra of $H_{\rm G}$ and $H_{\rm VP}$, the isomorphism (25) also provides a duality between $H^*_{\rm G}$ and $H^*_{\rm VP}$: $\Phi_{\rm d}\left(H^*_{\rm G}[\mu,\lambda]\right) = H^*_{\rm VP}[\mu,\lambda]$.

IV. CLASSICAL DUALITY

A. General ideas

The bond-algebraic approach to dualities in classical statistical mechanics can be performed in two ways. The first way exploits the well-known relationship between the partition functions of classical problems in $D = d + 1$ dimensions and quantum Hamiltonian problems in d. Using the closely related Suzuki–Trotter–Lie (STL) decomposition for quantum lattice models[10], this quantum-classical mapping takes the general form

$$\mathcal{Z}(K) = \text{Tr}\, \mathrm{e}^{-H[\lambda]} \tag{29}$$

where $\mathcal{Z}(K)$ stands for the path integral/partition function, and the classical, K, and quantum, λ, couplings are typically connected by non-linear functional relationships. Then the dualities between the quantum Hamiltonian can be translated into the corresponding dualities between the classical partition functions.

The second way is to associate bond algebras to partition functions of classical models through the transfer matrix. For example, row-to-row transfer matrices express the partition function as

$$\mathcal{Z} = \text{Tr}\left(T_1 \ldots T_s\right)^N, \tag{30}$$

where N is an integer related to the number of sites in one of the lattice directions, and $T_1, \cdots, T_s$ contain information about the directions transverse to the time direction. The general arguments concerning the additive bond structure of quantum Hamiltonians can be repeated verbatim for transfer matrices, with the only difference that transfer matrices display a multiplicative rather than additive bond structure:

$$T_i = \prod_\Gamma T_{i\Gamma} \tag{31}$$

The $\{T_{i\Gamma}\}_{\Gamma,i=1,\ldots,s}$ are now the bonds of interest, and the bond algebra proceeds as before. The duality is an isomorphism between the bond algebra $\mathcal{A}\{T_{i\Gamma}\}_{\Gamma,i=1,\ldots,s}$ and $\mathcal{A}\left\{T^D_{i\Gamma}\right\}_{\Gamma,i=1,\ldots,s}$, generated by a set of dual bonds

$\{T^D_{i\Gamma}\}_{\Gamma,i=1,\dots,s}$. The dual transfer matrices $T^D_i = \mathcal{U}_{\rm d} T_i \mathcal{U}_{\rm d}^\dagger = \prod_\Gamma T^D_{i\Gamma}$ will define, through Eq.(30), a partition function $\mathcal{Z}^D$ that may look very different from $\mathcal{Z}$. However,

$$\mathcal{Z}^D = \mathrm{Tr}\left(T^D_1 \cdots T^D_s\right)^N = \mathrm{Tr}\left(\mathcal{U}_{\rm d} T_1 \cdots T_s \mathcal{U}_{\rm d}^\dagger\right)^N = \mathcal{Z}. \tag{32}$$

This relationship between partition functions is the classical bond-algebraic duality.

B. Duality by quantum-classical mapping

In this section, I illustrate the first method (classical duality through the STL decomposition) by considering the duality between the $D = 2$ dimensional XY model and the $D = 2$ solid on solid (SoS) model. Their partition functions are

$$\mathcal{Z}_{\rm XY} = \sum_{\{\theta_{\boldsymbol{r}}\}} \exp\left[\sum_{\boldsymbol{r}} \sum_{\nu=1,2} K\cos\left(\theta_{\boldsymbol{r}+\boldsymbol{e}_\nu} - \theta_{\boldsymbol{r}}\right)\right], \quad \theta_{\boldsymbol{r}} \in \mathbb{R},$$
$$\mathcal{Z}_{\rm SS} = \sum_{\{m_{\boldsymbol{r}}\}} \exp\left[\sum_{\boldsymbol{r}} \sum_{v=1,2} K\left(m_{\boldsymbol{r}+\boldsymbol{e}_\nu} - m_{\boldsymbol{r}}\right)^2\right], \quad m_{\boldsymbol{r}} \in \mathbb{Z}. \tag{33}$$

First, consider the $D = 2$ XY model whose Hamiltonian is

$$H_{\rm XY} = -J\sum_{\boldsymbol{r}} \sum_{\nu=1,2} \cos\left(\theta_{\boldsymbol{r}+\boldsymbol{e}_\nu} - \theta_{\boldsymbol{r}}\right). \tag{34}$$

It can be mapped to the $d = 1$ quantum rotor model[11] specified by the Hamiltonian

$$\begin{aligned} H_{\rm rotor}[\lambda] &= \sum_i \left(\frac{1}{2}{L_i}^2 - \lambda\cos\left(\theta_{i+1} - \theta_i\right)\right) \\ &= H_L + H_\theta, \quad L_i = -i\frac{\partial}{\partial\theta_i}. \end{aligned} \tag{35}$$

Using STL decomposition, the partition function of the quantum rotor model can be written as

$$\begin{aligned} \mathcal{Z}_{\rm rotor}(\lambda) &= \mathrm{Tr}\left(e^{-H_{\rm rotor}}\right) = \sum_{\theta_1} \langle\theta_1| e^{-H_{\rm rotor}} |\theta_1\rangle \\ &= \lim_{M\to\infty} \sum_{\theta_1,\cdots\theta_M} \langle\theta_1| e^{-H_L/M} e^{-H_0/M} |\theta_M\rangle \langle\theta_M| e^{-H_L/M} e^{-H_0/M} |\theta_{M-1}\rangle \cdots \langle\theta_2| e^{-H_L/M} e^{-H_0/M} |\theta_1\rangle \\ &= \lim_{M\to\infty} \sum_{\theta_1,\cdots\theta_M} \langle\theta_1| e^{-H_L/M} |\theta_M\rangle e^{-H_\theta(\theta_M)/M} \langle\theta_M| e^{-H_L/M} |\theta_{M-1}\rangle e^{-H_\theta(\theta_{M-1})/M} \cdots \langle\theta_2| e^{-H_L/M} |\theta_1\rangle e^{-H_0(\theta_1)/M}. \end{aligned} \tag{36}$$

$|\theta_j\rangle = |\theta_{j,1}\rangle|\theta_{j,2}\rangle\cdots|\theta_{j,N}\rangle$ is the N-spin angle eigenstate. From Eq.(36), one should apply periodic boundary conditions (PBC) along the temporal direction: $|\theta_{M+1}\rangle = |\theta_1\rangle$. By inserting a complete set of angular momentum eigenstates $|m\rangle$, the matrix element of $e^{-H_L/M}$ is calculated to be

$$\begin{aligned} &\langle\theta_{j+1}| e^{-H_L/M} |\theta_j\rangle = \sum_m \langle\theta_{j+1}| e^{-H_L/M} |m\rangle \langle m \mid \theta_j\rangle = \prod_k \sum_{m_k} \exp\left[im_k\left(\theta_{j+1,k} - \theta_{j,k}\right) - \frac{\mu}{2M} m_k^2\right] \\ &= \prod_k \sum_{m_k} \int_{-\infty}^{+\infty} \exp\left[2\pi i m_k \phi + i\left(\theta_{j+1,k} - \theta_{j,k}\right)\phi - \frac{\mu}{2M}\phi^2 d\phi\right] = \prod_k \sum_{m_k} \sqrt{\frac{2\pi M}{\mu}} \exp\left[-\frac{1}{2\mu}\left(2\pi m_k + \theta_{j+1,k} - \theta_{j.k}\right)^2 M\right] \\ &\approx \prod_k \sqrt{\frac{2\pi M}{\mu}} \exp\left[\frac{M}{\mu}\cos\left(\theta_{j+1,k} - \theta_{j,k}\right) - \frac{M}{\mu}\right] = \exp\left[-\frac{MN}{\mu} + \frac{N}{2}\ln\left(\frac{2\pi M}{\mu}\right) + \sum_k \frac{M}{\mu}\cos\left(\theta_{j+1,k} - \theta_{j,k}\right)\right]. \end{aligned} \tag{37}$$

The second line uses the Poisson summation formula, and the third line uses the Villain approximation. Finally,

$$\mathcal{Z}_{\rm rotor}\left(\lambda\right)=A\cdot\lim_{M\to\infty}\sum_{\{\theta_{j,k}\}}\exp\left[\sum_{jk}\frac{M}{\mu}\cos\left(\theta_{j+1,k}-\theta_{j,k}\right)+\frac{\lambda}{M}\cos\left(\theta_{j,k+1}-\theta_{j,k}\right)\right]=A\cdot\lim_{M\to\infty}\mathcal{Z}_{\rm XY}\left(\frac{M}{\mu},\frac{\lambda}{M}\right)$$
$$\to A\cdot\mathcal{Z}_{\rm XY}\left(\sqrt{\lambda}\right). \tag{38}$$

In the last line, $\delta\tau=1/M$ is set to the natural ultraviolet cut-off, the inverse of the Josephson plasma frequency $\hbar/\sqrt{\lambda}$, without changing the universality class[11].

Next, consider the $d=1$ quantum SoS model[7] with states labeled by integers $\{|m_i\rangle\}$.

$$H_{\rm qSS}=\frac{1}{2}\sum_i\left(-\lambda\left(R_i+R_i^\dagger\right)+\left(X_{i+1}-X_i\right)^2\right), \tag{39}$$

with

$$X|m\rangle=m|m\rangle,\quad R|m\rangle=|m-1\rangle,\quad R^\dagger|m\rangle=|m+1\rangle, \tag{40}$$

that satisfy

$$\left[X,R^\dagger\right]=R^\dagger,\quad [X,R]=-R,\quad RR^\dagger=I. \tag{41}$$

By a similar argument, one can connect this quantum model to the partition function of the $D=2$ classical SoS model

$$\mathcal{Z}_{\rm qSS}\left(\lambda\right)=\mathrm{Tr}\left(e^{-H_{\rm qSS}}\right)=B\cdot\mathcal{Z}_{\rm SS}\left(\frac{1}{2\sqrt{\lambda}}\right) \tag{42}$$

At last, the isomorphism

$$L_i\xrightarrow{\Phi_{\rm d}}\left(X_{i+1}-X_i\right),\quad \mathrm{e}^{\mathrm{i}(\theta_{i+1}-\theta_i)}\xrightarrow{\Phi_{\rm d}}R_{i+1} \tag{43}$$

establishes a duality between between $H_{\rm rotor}$ and $H_{\rm qSS}$: $\Phi_{\rm d}\left(H_{\rm rotor}[K]\right)=H_{\rm qSS}[K]$. Combining those relations, we finally get the duality between the classical XY model and the SoS model:

$$\mathcal{Z}_{\rm XY}\left(K\right)\xrightarrow{\Phi_{\rm d}}\mathcal{Z}_{\rm SS}\left(K^*\right),\quad K^*=\frac{1}{2K}. \tag{44}$$

The XY model's high-temperature phase corresponds to the SoS model's low-temperature phase, and vice versa.

C. Duality by transfer matrices

This section will use the transfer matrix method to establish the duality between the $D=3$ classical $\mathbb{Z}_p$ gauge and the VP model. Their (generalized) partition functions are given by

$$\mathcal{Z}_{\rm G}(K,L)=\exp\left[\sum_{m=0}^{p-1}\sum_{\boldsymbol{r}}K_m\sum_{\nu=1,2}\cos\left(m\Theta_{(\boldsymbol{r},\nu)}\right)+L_m\cos\left(m\Theta_{(\boldsymbol{r},3)}\right)\right], \tag{45}$$

$$\mathcal{Z}_{\rm VP}(\bar{K},\bar{L})=\exp\left[\sum_{m=0}^{p-1}\sum_{\boldsymbol{r}}\bar{K}_m\sum_{\nu=1,2}\cos\left(m\theta_{\boldsymbol{r}}-m\theta_{\boldsymbol{r}+\boldsymbol{e}_\nu}\right)+\bar{L}_m\cos\left(m\theta_{\boldsymbol{r}+\boldsymbol{e}_3}-m\theta_{\boldsymbol{r}}\right)\right]. \tag{46}$$

with

$$\Theta_{(\boldsymbol{r},3)}=\theta_{(\boldsymbol{r},1)}+\theta_{(\boldsymbol{r}+\boldsymbol{e}_1,2)}-\theta_{(\boldsymbol{r}+\boldsymbol{e}_2,1)}-\theta_{(\boldsymbol{r},2)},\quad \text{etc.} \tag{47}$$

The angle variables $\theta_{(\boldsymbol{r},1)}$ (for $\mathcal{Z}_{\rm G}$) or $\theta_{\boldsymbol{r}}$ (for $\mathcal{Z}_{\rm VP}$) are defined on the links or sites on the $D=3$ square lattice (see Figure.1), and take p discrete values $\theta=2\pi s/p, s=0,\cdots,p-1$. Both coefficients K_m and L_m are real and symmetric.

To determine the transfer matrix $T_{\rm G}$ of $\mathcal{Z}_{\rm G}$, we need to partially fix the gauge of the model by considering only configurations that satisfy the constraint $\theta_(r,3)=0$. Since any other configuration can be obtained from one satisfying this constraint by a gauge transformation, the restriction has no physical consequence as long as we only compute averages of gauge-invariant observables. Under these conditions,

$$\mathcal{Z}_{\rm G}(K,L)=\widetilde{N}_{\rm G}\,\mathrm{Tr}\left[T_{\rm G}(K,L)\right]^{N_3},$$
$$T_{\rm G}(K,L)=\mathrm{e}^{-H^1_{\rm G}[\mu]}\mathrm{e}^{-H^0_{\rm G}[\lambda]}. \tag{48}$$

with $\widetilde{N}_{\rm G}$ a counting factor introduced to compensate for the gauge-fixing condition, and N_3 is the number of sites in the third lattice direction. $H^0_{\rm G}[\lambda]$, $H^1_{\rm G}[\mu]$ are given in Eq.(27). The coupling constants are related by the

discrete Fourier transform (see Eq.(A7))

$$e^{\sum_{s=0}^{p-1} K_s \cos(m\theta_s)} = \frac{1}{p}\sum_{s=0}^{p-1} \cos(m\theta_s)\, e^{\sum_{l=0}^{p-1} \mu_l \cos(l\theta_s)}, \qquad L_m = \lambda_m. \tag{49}$$

Similarly, the classical VP partition function can be written as

$$\mathcal{Z}_{\rm VP}(\bar{K},\bar{L}) = \mathrm{Tr}\left[T_{\rm VP}(\bar{K},\bar{L})\right]^{N_3}, \qquad T_{\rm VP}(\bar{K},\bar{L}) = \mathrm{e}^{-H^1_{\rm VP}[\bar{\lambda}]}\mathrm{e}^{-H^0_{\rm VP}[\bar{\mu}]}. \tag{50}$$

with $H^0_{\rm VP}[\bar{\mu}]$, $H^1_{\rm VP}[\bar{\lambda}]$ given in Eq.(28), and the coupling constants are related by

$$e^{\sum_{s=0}^{p-1} \bar{L}_s \cos(m\theta_s)} = \frac{1}{p}\sum_{s=0}^{p-1} \cos(m\theta_s)\, e^{\sum_{l=0}^{p-1} \bar{\lambda}_l \cos(l\theta_s)}, \qquad \bar{K}_m = \bar{\mu}_m. \tag{51}$$

In both cases, H^0 describes the interaction within a lattice plane with constant r^3, while H^0 describes the interaction between neighboring lattice planes.

According to the discussion at the end of Section.III B, there exists an isomorphism $\Phi_{\rm d}$ that provides a duality between $H^*_{\rm G}$ and $H^*_{\rm VP}$. Section.III A shows that $\Phi_{\rm d}$ can be implemented as an isometry $\mathcal{U}_{\rm d}$ satisfying (18) and (19). Hence,

$$\begin{aligned}\mathcal{Z}_{\rm G}(K,L) &= \widetilde{N}_{\rm G}\,\mathrm{Tr}\left[T_{\rm G}(K,L)\right]^{N_3}\\ &= N_{\rm G}\,\mathrm{Tr}\left[\left(\mathrm{e}^{-H^1_{\rm G}[\mu]}\mathrm{e}^{-H^0_{\rm G}[\lambda]}\right)^{N_3} P_{\rm GI}\right]\\ &= N_{\rm G}\,\mathrm{Tr}\left[\left(\mathcal{U}_{\rm d}\mathrm{e}^{-H^1_{\rm G}[\mu]}\mathcal{U}_{\rm d}^\dagger\mathcal{U}_{\rm d}\mathrm{e}^{-H^0_{\rm G}[\lambda]}\mathcal{U}_{\rm d}^\dagger\right)^{N_3}\right]\\ &= N_{\rm G}\,\mathrm{Tr}\left[\left(\mathrm{e}^{-H^0_{\rm VP}[\mu]}\mathrm{e}^{-H^1_{\rm VP}[\lambda]}\right)^{N_3}\right]\\ &= N_{\rm G}\,\mathrm{Tr}\left[T_{\rm VP}(K^*,L^*)\right]^{N_3} = N_{\rm G}\mathcal{Z}_{\rm VP}(K^*,L^*).\end{aligned} \tag{52}$$

And the dual couplings defined by the discrete Fourier transform

$$e^{\sum_{s=0}^{p-1} K^*_s \cos(m\theta_s)} = \frac{1}{p}\sum_{s=0}^{p-1} \cos(m\theta_s)\, e^{\sum_{l=0}^{p-1} K_l \cos(l\theta_s)} \tag{53}$$

similarly for L. In particular, one can set $L = K$ since they transform under the duality in the same way.

This duality can be simplified in some special cases, such as $p = 2, 3, 4$. Indeed, the summation over m in (IV C) can be dropped, and only the $m = 1$ term will survive. To see this, the K_0 term only contributes to an overall constant, irrelevant to the statistical properties. Due to the symmetry $K_1 = K_{p-1}$ and $\cos(\theta_{\boldsymbol{r}}) = \cos((p-1)\theta_{\boldsymbol{r}})$, these two terms can be combined into the K_1 term. At last, when $p = 4$, one can show that if $K_2 = 0$, then $K^*_2 = 0$ as well by the formula (53). Therefore, we can consistently only keep the K_1 term in both partition functions, which reduce to the conventional partition functions of the classical $\mathbb{Z}_p$ gauge and the VP model[7]. To be more explicit, consider the case $p = 2$. Then, (IV C) simplifies to

$$\begin{aligned}\mathcal{Z}_{\rm G}(K) &= \exp\left[K\sum_{\boldsymbol{r}}\sum_{\mu}\cos\left(m\Theta_{(\boldsymbol{r},\nu)}\right)\right]\\ &= \sum_{\{\sigma_{(\boldsymbol{r},\mu)}\}}\exp\left[K\sum_{\boldsymbol{r}}\sum_{\mu>\nu}\sigma_{(\boldsymbol{r},\mu)}\sigma_{(\boldsymbol{r}+\boldsymbol{e}_\mu,\nu)}\sigma_{(\boldsymbol{r}+\boldsymbol{e}_\nu,\mu)}\sigma_{(\boldsymbol{r},\nu)}\right],\\ \mathcal{Z}_{\rm VP}(\bar{K}) &= \exp\left[\bar{K}\sum_{\boldsymbol{r}}\sum_{\mu}\cos\left(m\theta_{\boldsymbol{r}} - m\theta_{\boldsymbol{r}+\boldsymbol{e}_\mu}\right)\right]\\ &= \exp\left[\bar{K}\sum_{\boldsymbol{r}}\sum_{\mu}\sigma_{\boldsymbol{r}}\sigma_{\boldsymbol{r}+\boldsymbol{e}_\mu}\right]\end{aligned} \tag{54}$$

Thus, the duality reduces to the well known duality between the $D = 3$ Ising model and $\mathbb{Z}_2$ gauge model. One can regard the duality in (52) as its generalization in the elementary degrees of freedom from the Pauli group to the Heisenberg-Weyl group. To the best of my knowledge, this derivation from the bond algebraic approach is first explicitly laid out in this paper.

V. CONCLUSION

I have discussed briefly the bond algebraic approach to both the quantum and classical dualities in this paper. Since the bond algebra is model specific, it can provide more varieties of duality even for the same model. It's also efficient in eliminating the gauge constraints and reduce the model to a gauge symmetry free model. This approach also have more applications than what's covered in this paper. For example, it can be applied to study the duality in quantum field theory, which has a continuous degrees of freedom. The self-dual field theories that display the phenomenon of dimensional reduction are of prime interest in the theory of topological quantum order. Besides, as mentioned before, the fermionization techniques such as the Jordan–Wigner transformation can also be analyzed through the prism of bond algebras, which illustrates that it is not at all necessary to consider non-local transformations. For further information about the bond algebraic approach, one can refer to the paper by Cobanera *et al.*[6, 7].

Appendix A: Weyl group algebra

In this appendix, I review some basic facts about the Weyl group algebra[12]. It's a finite group with generators U and V characterized by the relationships

$$VU = \omega UV, \quad V^p = 1 = U^p, \tag{A1}$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/p}$ is a p th root of unity. Eq.(A1) completely determines the finite-dimensional irreducible representations of U and V. A p-dimensional unitary matrix representation is given by

$$V = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad U = \mathrm{diag}\left(1, \omega, \omega^2, \ldots, \omega^{p-1}\right). \tag{A2}$$

The matrix V is called the fundamental circulant matrix and the matrix U is the phase matrix.

The Weyl group algebra admits a unitary automorphism Φ_{d} as a discrete Fourier transform. A direct calculation reveals that the unitary and symmetric Fourier matrix $F^{\dagger}_{mn} = \omega^{mn}/\sqrt{p}$ maps

$$\Phi(U) = V^{\dagger} = \mathrm{FUF}^{\dagger}, \qquad \Phi(V) = U = \mathrm{FVF}^{\dagger}. \tag{A3}$$

Any circulant matrix can be expanded in terms of the powers of V, such that

$$\mathrm{e}^{\sum_{m=0}^{p-1} a_m V^m} = \sum_{m=0}^{p-1} b_m V^m. \tag{A4}$$

In physical applications, the a_m are Hermitian-symmetric, $a_{p-m} = a_m^*$ (to guarantee that $\sum_{m=0}^{p-1} a_m V^m$ is a Hermitian operator), and the b_m are real and positive. Thus, it is convenient to assume that both sets of coefficients are real and satisfy $a_{p-m} = a_m, b_{p-m} = b_m$. Then by using F to diagonalize V, $FVF^{\dagger} = U$, and that $\mathrm{Tr}\left(U^{m\dagger}U^n\right)/p = \delta_{m,n}$, one can show that

$$\begin{aligned} b_m &= \frac{1}{p}\sum_{s=0}^{p-1} \cos\left(\frac{2\pi m s}{p}\right) \mathrm{e}^{\sum_{l=0}^{p-1} a_l \cos(2\pi l s/p)}, \\ a_m &= \frac{1}{p}\sum_{s=0}^{p-1} \cos\left(\frac{2\pi m s}{p}\right) \ln\left(\sum_{l=0}^{p-1} b_l \cos\left(\frac{2\pi l s}{p}\right)\right). \end{aligned} \tag{A5}$$

Since b_m is periodic and symmetric, it can be expanded as

$$b_m = \mathrm{e}^{\sum_{s=0}^{p-1} c_s \cos(m\theta_s)}, \qquad \theta_s = 2\pi s/p. \tag{A6}$$

then the relationship (A4) and (A5) becomes the discrete Fourier transform

$$\begin{aligned} \mathrm{e}^{\sum_{m=0}^{p-1} a_m V^m} &= \sum_{m=0}^{p-1} \mathrm{e}^{\sum_{s=0}^{p-1} c_s \cos(m\theta_s)} V^m, \\ \mathrm{e}^{\sum_{s=0}^{p-1} c_s \cos(m\theta_s)} &= \frac{1}{p}\sum_{s=0}^{p-1} \cos\left(m\theta_s\right) \mathrm{e}^{\sum_{l=0}^{p-1} a_l \cos(l\theta_s)}. \end{aligned} \tag{A7}$$

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