“Let us not forget this: when I raise my arm, my arm goes up. And now the problem arises: what is left over if I subtract the fact that my arm goes up from the fact that I raise my arm?” (para.621) I suppose we could try to take Wittgenstein straight here: the problem is to find the remainder. But more likely Wittgenstein is raising a problem about the first problem; how is it that even before beginning to look for the remainder we feel sure that it must be there, that there is a definite something to be looking for?

This meta-problem is discussed by Robert Jaeger in a 1973 Phil Review paper called “Action and Subtraction.” He starts by noting that semantic notions have in some cases arithmetical analogues. Take for instance conjunction and equivalence:

   conjunction is like addition and ...logical equivalence is like equality (328).

Now in arithmetic, we know, addition and equality let us define subtraction:

   (1) \( a - b \) is the \( r \) such that \( r + b = a \).

So one could be forgiven for assuming that a semantic analogue of subtraction would be likewise definable from conjunction and equivalence:

   (2) \( A - B \) is the \( R \) such \( R \ & B \) is equivalent to \( A \).
Just as (1) leaves no doubt whatever about which number is the numerical difference between \( a \) and \( b \) – it’s pretty close to an algorithm for finding that difference – it would seem that (2) ought to leave no doubt whatever about which proposition is the logical or semantical difference between \( A \) and \( B \). The problem is that, these assurances notwithstanding, it is in fact extremely difficult to identify that difference in in examples like Wittgenstein’s, and it is far from clear that the difference is well-defined.

That according to Jaeger is the problem, or one problem, Wittgenstein is raising, and here is Jaeger’s proposed solution (which I must say strikes me as awfully plausible):

The question “What is left over?”...presupposes ....that there is exactly one statement with certain logical properties (321).

This presupposition is false:

whereas there is exactly one number \( r \) such that \( r+2=5 \), it is not the case that there is exactly one statement \( R \) such that “\( R \) & my arm goes up” is logically equivalent to “I raise my arm” (328).

Questions with false presuppositions can’t be answered, so that is why we have difficulty saying what more there is to my raising my arm than my arm going up.

This might seem like old news, but if so then the news hasn’t penetrated to every outpost. For the point, as Jaeger himself observes, also applies to a lot of questions besides Wittgenstein’s, including some that philosophers continue to take seriously. Two examples he mentions are
'What is left over if I subtract the fact that $P$ from the fact that I know (see, remember, am pleased, am sorry) that $P$?' and "What is left over if I subtract the fact that there is an $F$ from the fact that I see (hear, perceive, observe, am aware of) an $F$?" 

These ought to be recognizable as questions that Tim Williamson works hard to discredit in *Knowledge & Its Limits*, which suggests that they were still being pursued as recently as 2000. Examples he doesn’t mention but might have are the attempt by narrow content theorists to disentangle the “organismic contribution” (Dennett’s phrase) to a wide psychological state from external factors like the watery stuff’s chemical composition; the attempt by neo-Lockeans about personal identity to develop a notion of quasi-memory that doesn’t presuppose identity; in the attempt by certain moral theorists to single out as the proper object of evaluation an act of will so pure and internal that considerations of moral luck can’t get a grip.

One point Jaeger is making about all these cases is that although one feels there must be a remainder, there is really no must about it. Even that, though, might be conceding too much; for to say there is no must about suggests that we know what a remainder fact would be, it’s just that there

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1 Jaeger anticipates Williamson’s claim that knowledge among other states is prime, not a conjunction of something inner with something outer. Note that even Williamson allows himself the occasional appeal to subtraction, e.g., when he characterizes belief as botched knowledge.

2 Williamson says the questions presume that each mental state is composite – the conjunction of an internal factor with an external one – when in fact they are prime – not decomposable in this way because an inextricable mix of the internal and the external.
happens not to be one in this case. And Jaeger might be read as suggesting that the very notion of a remainder fact is unclear, perhaps irremediably so. It’s not that we know what we’re looking for but can’t find it; we don’t even know what we’re looking for.

So, to review. Wittgenstein asked what is left when you subtract my arm’s going up from my raising it. The idea of logical or semantic subtraction – what happens when you subtract one proposition or fact or circumstance from another – has never recovered from Wittgenstein’s example, and later examples of the same sort. The question “what is left over?” seems misconceived and the subtraction idea with it.

The story might seem over at this point, but it isn’t, for mystifying as subtraction is in the cases discussed, there are other cases where it seems to make perfect sense. What is left of snow’s being cold and white when you subtract the fact that snow is cold? That’s easy: what is left is that snow is white. What is left of the fact that both my children play soccer, when we subtract that I have two children? The fact that all my children play soccer. These examples are a bit artificial, but consider the following (some are taken from Andre Fuhrmann, “When Hyperpropositions Meet”):

These two polygons are similar, that is, congruent but for the requirement of being the same size.

Justification is what remains when you subtract true belief from knowledge.

A gratin is a quiche that is not baked in a shell.
“Copper is an insulator” is lawlike, it falls short of being a law only in respect of truth.

Having a visual experience as of BLAH is seeing that BLAH, stripped of any implication that BLAH is the case.

Attempted murder is murder, except the victim need not die.

[Will Rogers: It’s not what he doesn’t know that bothers me, it’s what he does know that just isn’t true.]

If there were a problem about subtraction as such, then these sorts of statements would be automatically nonsense. And that is certainly not how they strike us. They’re not automatically sensical, perhaps, but some of them do seem to make sense and of those some could turn out to be true.

I see no point in insisting, then, that subtracting B from A is never a meaningful operation. There is room, however, to argue about the larger significance of the operation. Is there anything genuinely “new” going on, that semantical theory has overlooked, or is it something familiar presented in unnecessarily mysterious guise?

The analogy with arithmetic suggests semantic subtraction is not anything new. Why is that? Well, subtraction in arithmetic is reducible to addition and polarity reversal; to subtract b from a is no different from adding a to b’s “opposite,” namely minus-b. One might then expect that to semantically subtract B from A was the same as adding A to B’s “opposite,” its negation. If

\[ (3) \ a - b = a + (-b), \]
then one could be forgiven for thinking that

(4) $A - B = A \& \sim B$.

But this is a mistake. Suppose we tried to treat lawlikeness, say, as the conjunction of lawhood with the negation of truth:

$S$ is lawlike iff (i) $S$ is a law, and (ii) $S$ is not true.

If that were the claim then for a statement to be lawlike would be ipso facto for it to be untrue. And that was not the idea at all; a lawlike statement is supposed to be a candidate for truth, rising to the level of a law if and only if its candidacy succeeds. More generally $A - B$ is not supposed to deny that $B$ obtains, it’s just supposed not to require that $B$ obtains.

That said, in some of the cases the subtrahend $B$ does seem to be denied. A statement may still be lawlike if it turns out to be true, but my poisoning Jones cannot hope to remain a case of attempted murder if it causes Jones to die, and this concoction of eggs and cheese does not remain a gratin if it’s baked in a shell. The question then arises, does subtraction in these other cases involve anything more than conjunction with $B$’s negation?

The answer is again NO, because if $A$ implies $B$, which one assumes it does or why are we trying to subtract $B$ from it, then $A \& \sim B$ is a flat-out contradiction. This is so whether the overall statement implies $\sim B$ (as attempted murder implies the negation of fatal) or not (as lawlikeness fails to imply the negation of truth). What makes the cases different is not that the subtraction takes one form – cancellation – in the definition of lawlikeness – and another form – assertion of the opposite –
in the definition of attempted murder. Cancellation occurs in both cases, else attempted murders would require the impossible combination of murder with nobody dying. The difference is that one stops at cancellation in the definition of “lawlike” (so as to leave it an open question whether lawlike statements are true) whereas one goes on to replace it with the opposite requirement in the definition of attempted murder (so as not to leave it an open question whether attempted murder is fatal). To put the point formally, consistency requires us to first whittle away the B aspect of A before we can think of adding on ~B on as a second conjunct.

It seems then that subtraction is doing special semantic work for us in some cases. If that is right then we can’t just reject the idea wholesale. Instead we should seek an account of subtraction that clarifies why the operation sometimes misfires, why there is sometimes no such thing as the difference between one circumstance and another.

An initial proposal, already hinted at, would be that A – B exists just when there is an R such that A is (equivalent to) the conjunction of R and B. Certainly if you look at the “good” examples above, where subtraction is intuitively well-defined, such an R readily suggests itself. That both my children play soccer (that’s A) is the conjunction of (R) all my children playing soccer with (B) there being two of them. And clearly, that (A) snow is cold and white is the conjunction of (R) snow’s being white with (B) its being cold.

Unfortunately it is true also of the “bad” examples, where A - B is intuitively undefined, that an R can be found such that A is (equivalent to) the conjunction of R with B. My raising my arm (A) is the conjunction of (B) my arm’s going up with, say, my raising it, or if you prefer my raising it if it goes up. Consider
another paradigmatically “bad” example: one cannot subtract a thing’s being colored (B) from its being red (A), but its being red is nevertheless the conjunction of its being colored (B) with its being red, or if you prefer, its being colored if red.

OK, second try: A – B exists just when there is a unique R such that A is equivalent to the conjunction of R and B.

(5) R is A - B iff it uniquely satisfies A = R & B,

On the plus side, this stronger condition is not met in the “bad” cases just mentioned; as we just saw, there are lots of Rs which added to a thing’s being colored yield its being red. The trouble this time is that it true also of the “good” cases, where A – B is intuitively defined, that there are lots of Rs which conjoined with B yield something equivalent to A. To get snow’s being cold and white from its being cold, you can add snow’s being white, but you can also add something stronger – its being both white and cold – or if you prefer something weaker – snow’s being white if cold.

A fact we should not lose sight of, when thinking about the non-uniqueness problem, is that where there are various candidates Rᵢᵣ for the role of A - B – various Rs such that A is equivalent to Rᵢᵣ&B – their conjunction is also a candidate in the same sense. So perhaps instead of (5) what we want is

(6) R is part of A - B iff it satisfies A = R & B,

This can’t be quite right because A & B solves the equation for R, and we definitely don’t want the sum of A and B to be part of the difference between them. But A & B is independently objectionable on grounds of not being suitably independent of B – of not being, as I’ll say B-free. If we could cash out this
notion of B-freedom, we could define A - B as the sum of all B-free R, that is,

(7) R is part of A - B iff A ≡ R & B and R is B-free.

So our question boils down to this: what does it take for R to be free of any taint of B? This is a very hard question but luckily we don’t have to start from scratch. David Lewis has developed some machinery in “Relevant Implication” and “Statements Partly About Observation” that sheds non-trivial light on the notion of B-freedom.

One of the things Lewis does in these papers is set out a theory of subject matters. A subject matter for Lewis is a partition of the set of worlds, where two worlds are in the same cell if, intuitively speaking, they are exactly alike with respect to that subject matter. Take for instance the subject matter of the 19th century: worlds are indiscernible with respect to that subject matter iff the one’s 19th century is a duplicate of the other’s. This is a parts-based subject matter in the sense that worlds go into the same cell iff a certain spatiotemporal part of one duplicates the corresponding part of the other. But not all subject matters are parts-based. Take the subject matter of how many stars there are. One cell contains all the worlds with no stars; another contains all the worlds with one star; and so on. Clearly worlds can have the same number of stars without there being any spatiotemporal region in which one duplicates the other.

Now Lewis has a couple of reasons for being interested in subject matters. One has to do with the enduringly frustrating issue of aboutness. Lewis says a statement is wholly about a given subject matter if its truth-value never varies between worlds in the same cell. So the statement “There are infinitely
many stars” is wholly about the subject matter of how many stars there are, while “There are fewer stars than planets” is not wholly about that subject matter since worlds that agree in how many stars they contain might still disagree about whether the planets outnumber the stars. He tries to parlay this notion of wholly about into a serviceable notion of at least partly about, but the details that don’t concern us here.

Lewis is also interested in subject matters for the light they may shed on the nature of implication. He starts by defining a notion of orthogonality. Two subject matters are orthogonal iff things being one way with respect to the first never prevents them from being at the same time any other way with respect to the second. A bit more explicitly however things may be with respect to the first subject matter, and however things may be with respect to the second, it is possible for things to be both of these ways in a single world. Completely explicitly,

\[(8) \ M \text{ is orthogonal to } N \iff \text{ every cell of } M \text{ intersects every cell of } N.\]

So, for instance, how many stars there are is orthogonal to how many planets there are iff the number of planets puts no limits on the number of worlds and vice versa. Whether S is the case is orthogonal to whether T is the case iff every combination of truth-values is possible, that is, there are worlds where both S and T are true, worlds where both are false, world where S is true and T false, and worlds where T is true and S false.

Now Lewis uses orthogonality, or rather the lack of it, which he calls connectedness, in an attempt to elucidate relevant implication. P relevantly implies Q, the thought is, if P implies Q and P is relevant to Q. P is relevant to Q, Lewis suggests, if
all of P’s subject matters are connected to all of Q’s subject matters. I on the other hand want to use orthogonality to explain what it means for R is free of any taint of B, in the way it would need to be to be a candidate for the role of A - B.

One obvious idea to try here, and it’s close to what Jaeger does try, is to say that P is Q-free iff whether P is the case is orthogonal to whether Q is the case.

(9) P is Q-free iff whe(P) ⊥ whe(Q)

As in effect already noted, this orthogonality comes to the claim that P is logically independent of Q in the strong sense that there are no implication relations between P and its negation, on the one hand, and Q and its negation, on the other. I call (9) the simple theory of freedom.

But the simple theory does not work. An initial worry that I won’t dwell on is that it makes freedom symmetrical, that is, P is Q-free iff Q is P-free, thereby making subtraction symmetrical too, that is, C = A - B iff B = A - C. Subtraction seems to me quite clearly asymmetrical – lawlikeness may be lawfulness minus truth, but truth is not lawfulness minus likeliness – I don’t want to dwell on that now. My objection to (9) is two-fold; in some ways it doesn’t ask enough, in others it asks too much.

Why do I say (9) asks for too much? Imagine Alice and Bert have the following conversation just as the party is starting:

Alice: Has anyone arrived yet?
Bert: Some people have arrived, but no one I like.
Alice: How do you know no one you like?
Bert: Cuz I checked all the rooms.
Alice: Including the basement?
Bert: Uh oh, I guess I haven’t checked all the rooms. Until I check, better take back “the no one I like” part.

It seems pretty clear that what remains of Bert’s original conjunctive claim when he cancels the second conjunct is just the first conjunct: some people have arrived. But Some people have arrived is not in the relevant sense independent of No one nice has arrived; for two subject matters are orthogonal only if all four assignments of truth-value are possible – TT, TF, FT, and FF – and these two statements cannot be false together, since if no one has arrived, then trivially no one nice has arrived.

This suggests (what seems anyway plausible) that for P to be free of Q, it is not important that the two be capable of agreeing in truth-value. On the contrary, agreement in truth value is just what you’d expect if P and Q said much the same thing. We’ll come back to this after considering the simple theory’s opposite problem, that it makes freedom too cheap and easy.

Suppose I tell you that your dog Rex barks too much (call that A) and you raise a doubt about this. I reply, well, your dog is Rex, right (that’s R), and Larry just told me that Rex barks too much (B). You reply that Larry is a fool and was in any case talking about some other dog. So I take back B, the part about Rex barking. What is left?

On the face of it nothing is left but the first conjunct R, that your dog is Rex. Certainly I’m not still accusing your dog of barking too much! But the simple theory says I am; I have not withdrawn that original complaint. This is because, on the one hand, $A = “Your dog Rex barks too much”$ is equivalent to $R^* =$
“Your dog barks too much” conjoined with $B = \text{“Rex barks too much”}$ And on the other hand there are no necessary connections between the conjuncts, between your dog’s barking too much and Rex’s doing so (after all, you could have had a different dog). And that’s all it takes, according to the simple theory for your dog barking too much to be not just a consequence of what I said (that your dog Rex barks too much), but a consequence free of any taint of Rex barking too much, and so part of what remains when we subtract his barking too much. And again, that seems like leaving too much. If someone says, Rex is your dog and Rex barks too much, and then withdraws the part about Rex barking too much, one wouldn’t take them to still be committed to your dog’s barking too much.

So here is a respect in which the simple theory doesn’t go far enough: it’s not enough to require that certain combinations of truth-value be possible, the reasons for the truth-value matter too. Given that your dog is Rex, for your dog to do something is in part for Rex to do so (in part because Rex also has to be your dog). That it could in some far-away world have been some other dog of yours that was barking too much is well and good but not enough to eliminate the dependence in this world of your dog’s having a property on Rex’s having it.

Time now to stand back and draw some morals. The moral we draw from our first complaint about the simple theory is that it makes no difference whether $P$ and $Q$ can agree in truth-value; what matters to the kind of independence we’re after, namely that $P$ shouldn’t borrow too much from $Q$, is disagreement in truth-value. All that matters in the truth-value department is that $Q$ can be true when $P$ is false and vice versa. That’s what shows or begins to show that $P$ is staking out new ground. So, definition:
(10) Suppose \( M \) and \( N \) are subject matters such that \( P \) is wholly about \( M \) and \( Q \) is wholly about \( N \). As always, \( M \) is orthogonal to \( N \) iff every cell of \( M \) intersects every cell of \( N \). \( M \) is unaligned with \( N \) (in symbols \( M \uparrow N \)) iff every cell of \( M \) where \( P \) is true intersects every cell of \( N \) where \( Q \) is false and vice versa.

The requirement should never have been that \( \text{whe}(P) \) was orthogonal to \( \text{whe}(Q) \), but only that \( \text{whe}(P) \) was unaligned with \( \text{whe}(Q) \).

The moral we draw from our second criticism is that while unalignment may be necessary for \( Q \)-freedom, it is not enough that \( P \)’s falsity is compatible with \( Q \)’s truth and vice versa; for we’ve got that much already with \( P = \) your dog’s barking too much and \( Q = \) Rex’s barking too much, and this is not a case where \( P \) is \( Q \)-free. It seems that it is not just the fact of \( P \)’s falsity but the reasons for it that have to be independent of \( Q \)’s truth. Supposing your dog doesn’t in fact bark, at least one of the reasons for this is that Rex doesn’t bark, which is clearly not compatible with \( Q \)’s truth = Rex’s barking too much. It’s because the reason for your dog’s not barking too much conflicts with Rex’s barking too much that this is not a case of \( Q \)-freedom. (Mutatis mutandis if your dog does bark too much; the reasons for this include that Rex barks too much, which is incompatible with \( \sim Q = \) Rex’s not barking too much.)

So, where does this leave us? The idea that suggests itself when we take both morals into account is that

(11) \( P \) is \( Q \)-free iff the reasons why \( P \) is false (if it is) are compatible with \( Q \) being true, and the reasons why it is true (if it is) are compatible with \( Q \) being false.
Note it follows from this that whether \( P \) is \( Q \)-free is not a function of the two statements or propositions taken alone. Their truth-values matter too, and even the reasons for their truth-values. Given the role of \( Q \)-freedom in the definition of subtraction, the very identity of \( A - B \) may depend on truth-values. This may seem initially strange so let me give an example where truth-value makes a difference. Suppose \( A \) is “I have the magic beans in my pocket” and \( B \) is “There are magic beans.” A candidate \( R \) is “I have at least one magic bean in my pocket.” Is this \( B \)-free? Yes, if \( R \) is false; for in that case we ask whether the reason for its falsity – that my pocket is empty – are compatible with there being magic beans somewhere or other. The answer is yes and \( R \) is to that extent part of the difference between \( A \) and \( B \). This is nice because it explains why “I have the magic beans in my pocket” still seems false even bracketing the issue of whether there really are magic beans.

OK, but suppose now that the Jack story is true, and I really do have the magic beans in my pocket. Now to assess \( B \)-freedom we ask whether the reasons for its truth – that I’ve got \( x \), \( y \), and \( z \) in my pocket and they’re magic beans – are compatible with \( B \)’s falsity – with there being no such thing as magic beans. The answer this time is NO; so if \( R \) is true then it is not part of the difference between \( A \) and \( B \). This explains why, although “I have the magic beans in my pocket” retains its falsity if we bracket the issue of whether there really are such things, it doesn’t retain its truth if we try to bracket that issue, indeed it becomes ill-defined. If someone says, “What is left over if you subtract from your having the magic beans in your pocket any implication of there being magic beans?” the answer is “I have some magic beans in my pocket” if my pocket
is empty and “That’s a nonsensical question” if my pocket contains magic beans.

The proposed account of Q-freedom talks about the reasons why P is true or false. I suggest treating this a new subject matter: over and above the subject matter of whether P is or is not the case, there’s a subject matter of why P is or is not the case.

(12) why(P) is the partition induced by the “P is true (or false) for the same reason” relation on worlds.

The new subject matter is a refinement of whe(P) because worlds where P is true (false) for the same reason are ipso facto worlds in which P has the same truth-value. into cells whose members have P true for the same reason, and the worlds where P is false into cells whose members have P false for the same reason.

I am not going to be saying much of substance about the partition which serves as the subject matter of why P is or is not the case; we’re going to have to content ourselves with the formal features already mentioned and with a hopefully not too counterintuitive treatment of examples. I don’t think there is anything unusual here. I am using the same divide and conquer strategy that Lewis used in his work on counterfactuals and de re modality. He first tries to exhibit the truth-values of these statements as a function of the relevant parameter, similarity or counterpart relation. Only later does he come back around to ask what values the parameter should take to give the right sentence-level results, and to consider how it acquires the value that he needs it to have. Like Lewis in the earliest work, we’ll be sticking mostly to Job One.
So: let’s say we’ve been given the subject matter for each statement \( P \) of why \( P \) is or is not the case. There’s one cell of this subject matter that particularly interests us, namely the one containing us, or rather our world. If \( P \) is false in our world, then this cell can be characterized as the set of worlds where \( P \) is false with its actual falsity-maker; otherwise it’s the set of worlds in which \( P \) is true with its actual truth-maker. Either way, the set can be considered a subject matter; let’s call it \( \text{why } P \text{ is or is not actually the case}, \text{why}_\mathcal{S}(P) \) for short. It’s an unusual subject matter in two respects. It’s partial, since defined only on worlds where \( P \) has the same truth-value as here in actuality, and indeed only on a proper subset of those. And it’s a degenerate subject matter in that it has only the one cell. But it’s a subject matter nevertheless and a subject matter I use it to define freedom as follows.

\[
(13) \ P \text{ is } Q\text{-free iff: } \text{why}_\mathcal{S}(P) \uparrow \text{whe}(Q).
\]

If you track back through the definitions, you’ll see that for \( \text{why}_\mathcal{S}(P) \) to be unaligned with \( \text{whe}(Q) \) is just this: whatever \( P \)'s actual truth-value may be, it could have had that truth-value for the same reason even if \( Q \) had the opposite truth-value.

Now we come back around finally to the issue that concerns us, namely, how to define for two statements \( A \) and \( B \) the result of subtracting \( B \) from \( A \). The proposal to a first approximation is that

\[
(14) \ A - B \text{ is the sum-total of } A\text{'s } B\text{-free consequences.}
\]

(This is really no different from (7) above, it’s just that now we have some idea what the key term “\( B\)-free” means.) I say to a first approximation because, as stressed above ((5)-(7)), \( A - B \) should be something that combines with \( B \) to yield \( A \). (Jaeger:
“It is part of the meaning of ‘subtract’ that the minuend (whether a number, a set, or a statement) should be identical with the result of adding the subtrahend and the remainder” (“Logical Subtraction and the Analysis of Action,” 142). Call that the reversibility constraint:

(15) If we subtract \( B \) from \( A \) and then add \( B \) back again, the result should be \( A \) or equivalent to it \((A - B) \& B = A\).

There’s nothing in (14) to guarantee that the reversibility constraint is satisfied. After all, \( A \) might not have much in the way of \( B \)-free consequences, in which case combining \( B \) with what it has may not get you all the way back up to \( A \) again. So we need to qualify (14) a bit:

(16) \( A - B \) is the sum-total \( R \) of \( A \)'s \( B \)-free consequences, provided that \( R \& B \) imply \( A \)

This may seem a detail but it’s crucial if we want to understand why subtraction is sometimes well-defined and sometimes ill-defined. I suggest that one important reason for \( A - B \) to be undefined is that \( A \) does not have enough \( B \)-free consequences to meet the reversibility condition.

Against this it might be said that \( A \) is bound to have enough \( B \)-free consequences for reversibility to hold, since all one needs for reversibility is the extremely weak consequence that \( B \supset A \). Some say this material conditional just is the difference:

it seems reasonable to stipulate that if there are several different propositions whose conjunction with \( B \) is \( A \), then the least (i.e., the weakest) of these shall be considered the difference. This additional stipulation [along with (i) \( A \)
implies $\neg A \land B$, (ii) $(A \land B) \land B$ implies $A$, (iii) $A \land B$ does not imply $B$ (iv) $B$ does not imply $A \land B$] determines that logical subtraction be the converse of material conditionalization (Hudson, “Logical Subtraction,” 131).

But although Hudson is right that the conditional satisfies (iii) – $A \land B$ can be true with $B$ false – it doesn’t satisfy the slightly stronger requirement imposed by $B$-freedom, namely that the reasons $B \supset A$ is true (suppose it is) are compatible with $B$ being false. For the conditional might be true because its consequent $A$ is true. Is this compatible with $B$ being false? Not if $A$ implies $B$, which it always does in this paper. What makes it true that cherries are red if colored? It seems highly relevant that cherries are red. And this is not compatible with cherries’ not being colored, as it would have to be for the conditional to be free of any taint of their being colored.

I should stress that these considerations don’t show that reversibility is unattainable in some cases; they do show however is that it’s not automatic. I conjecture that the reason $\text{red} \land \text{colored}$ and raised my arm - the arm went up seem undefinable is that the minuend has very few consequences that are free of the subtrahend. Not much follows from a thing’s being red that is without prejudice to the issue of whether it’s colored. A little bit follows from my raising my arm that’s without prejudice to my arm’s going up – that I intended my arm to go up, for example – but not enough to imply that if my arm went up, then I raised my arm. For that you’d need something more like I intended my arm to go up and the intention was effective; but the reasons this is true are again not without prejudice to the issue of whether my arm goes up. So, proposal:
(17) $A - B$ is undefined if $A$ lacks $B$-free consequences strong enough to combine with $B$ to yield $A$, or more simply $A$ lacks $B$-free consequences strong enough to imply $B \supset A$.

This may bear on an issue raised long ago by Strawson. Strawson famously observed that if someone produced the words

The King of France is bald,

we would be apt to say that “the question of whether his statement was true or false simply did not arise, because there was no such person as the King of France” (1950, 12). But he notices that our intuitions sometimes go the other way: we find the sentence evaluable even though “there just is no such particular item at all” as the speaker purports to be talking about:

Suppose, for example, that I am trying to sell something and say to a prospective purchaser The lodger next door has offered me twice that sum, when there is no lodger next door and I know this. It would seem perfectly correct for the prospective purchaser to reply That's false, and to give as his reason that there was no lodger next door. And it would indeed be a lame defense for me to say, Well, it's not actually false, because, you see, since there's no such person, he question of truth and falsity doesn't arise (1954, 225).

Likewise if I tell you I had breakfast with the King of France, it would seem perfectly correct for you to reply That's false, and it would be a lame defense for me to say, it’s not actually false, the question of truth or falsity doesn’t arise. Here is a proposal for what is going on in these cases:
(18) Suppose $S$ semantically presupposes $\pi$ and $\pi$ is false. $S$ strikes us as unevaluable iff $S - \pi$ is ill-defined, that is, $S$ lacks $\pi$-free consequences strong enough to imply $\pi \supset S$. It strikes us as evaluable iff $S - \pi$ is well-defined as $R$, and as true/false according to whether $R$ is true/false.

Let’s see if (18) can explain the intuitive difference between “The KoF is bald” and “The KoF is in this chair.” Note first that the latter does have a $\pi$-free implication $R$ strong enough to imply “France has a unique king $\supset$ The KoF is in this chair,” namely “A French king sits in this chair.” The latter is a $\pi$-free consequence because it is false for a reason compatible with $\pi$’s truth, viz. that the chair is empty. So we can explain both why “The KoF sits in this chair” strikes us evaluable and and why it strikes us as false.

So far so good, but there is still the question of why “The KoF is bald” seems in comparison empty and unevaluable. The claim will have to be that “The KoF is bald” does not have a $\pi$-free consequence strong enough to imply “France has a unique king $\supset$ The KoF is bald.” Well, “A French king sits in this chair” worked last time, why not “A French king is bald”? The difference is that “A French king sits in this chair” is false because the chair is empty, which is perfectly compatible with France’s having a unique king off to the side. “A French king is bald” however is false because France lacks a king, which is not compatible with France’s having a unique king off to the side.

But, the question will come, why isn’t the chair-sentence also falsified by France’s lack of a king? Let me give two answers to this, aimed at two audiences. The first is for Strawsonians. Strawsonians think the chair-sentence is gappy unless there is such a thing as the chair. To them I say, look, France’s lack of a king does not imply anything about a chair; in chairless worlds where France lacks a king “A French king sits in this chair” is not false
but gappy. To put it more precisely, the subject matter why it is true/false that a French king sits in this chair is supposed to be a refinement of whether it is true/false that etc, and that means “A French king is false” must have the same truth-value in every world of a given cell.3

The second answer is for Russellians. Russellians think the chair sentence is false when there is no chair. To them I say, even if the chair-sentence is false both in worlds where the chair is empty and worlds where there is no chair, it is not false for the same reason in those worlds. Given two candidate falsity-makers, one speaking to goings on throughout space and time, one speaking to goings on in a small bounded region, the second is other things equal preferable. If that is right then in our world “A French king sits in this chair” is false because the chair is empty. And as before, that the chair is empty puts no obstacles in the way of France’s having a unique king.

So much concerns the question of why the chair-sentence is falsified by a π-free circumstance – that the chair is empty – rather than the the π-unfree circumstance of France’s lacking a king. There is a complementary question that might be raised, namely, why is the baldness-sentence falsified by the π-unfree circumstance of France lacking a king, as opposed to the π-free circumstance of its lacking a bald king? The answer is that truth- and falsity-makers are subject to a proportionality constraint. If X and Y are candidate truth-makers and Y is obtained from X by complicating it in ways that do nothing to enhance its truthifying powers, then X is the preferred candidate other things equal.

“Someone lives in the White House” is true because Bush lives there, not that plus the fact that he’d rather be on the ranch. The

3 What you’d need to ensure that is that France lacks a king and there is such a thing as the chair. But this seems needlessly complicated compared to the simple fact that the chair is empty.
same applies to falsity-makers: there too gratuitous complications are disqualifying. Given that France’s lack of a king already falsifies “France has a bald king,” to move to its lack of a bald king is all pain with no gain.

So a good deal of philosophical tradition to the contrary, France does not need to have a king for “The KoF sits in this chair” to be properly evaluable as false – properly because even ignoring the presupposition failure, what remains is false. One can argue in a similar way that “The KoF does not sit in this chair” is properly evaluable as true, quite regardless of whether there is such an individual as the King of France. (“The KoF does not sit in this chair” π-free implies that all French kings do not sit in the chair which is true. It also implies that some French kings do not sit in the chair, but that is not π-free, since it is false due in part to France’s lack of a king, and France’s lack of a king is π-incompatible.)

I hope you find this suggestive, because it hooks up with a key assumption of the problem of negative existentials. That problem supposedly arises as follows:

(a) “The F is G” makes a claim only if there is such a thing as the F.
(b) “The F doesn’t exist” is true only if there is no such thing as the F.
(c) “The F doesn’t exist” is true only if it makes a claim.
(d) “The F doesn’t exist” is true only if a contradiction holds. (a)-(c)
(e) “The F doesn’t exist” can never be true. (d)

We’ve now seen reason to doubt premise (a): that there is no such thing as the F does not stop “The F is G” from making the claim (if there is one) obtained by cancelling the presupposition that there is exactly one F. Of course, it’s one thing to note the possibility in principle, another to make it work in practice. Let me sketch then how I think it could work, while admitting up front that complications arise that I don’t know how to deal with.
The question is, what is left over when one strips from the claim that the KoF doesn’t exist the presupposition apparently triggered by the initial description, viz. that he exists after all (or rather before all)? The theory says that it’s the \( \pi \)-free implications that are left, and in this case \( \pi \) says that France has exactly one king; so what is implied by “The KoF doesn’t exist” that is free of any taint of France having exactly one king? I’ll consider just two candidates for the role of \( \pi \)-free implication, namely, some French king doesn’t exist, and no French kings exist.

Is “some French king doesn’t exist” \( \pi \)-free? No, because the reason for its falsity is that there are no French kings to exist or not, and that there are no French kings is is \( \pi \)-incompatible. “No French kings exist” is true for the same reason “Some don’t” was false, viz France has no king. The question is whether this truth-maker is compatible with \( \pi \)’s falsity, that is, with France not having a unique king. The answer is that it is not only compatible, it entails that \( \pi \) is false.

So it looks as if the incremental content here is that all French kings fail to exist. I suggest very tentatively that it is the truth of this incremental content that leads us to judge “The KoF does not exist” true despite the superficially self-undermining character that makes an attribution of truth seem problematic. It is judged true for the same sort of reason as “The KoF is not to be found in this chair” is judged true. Confession: I haven’t explained and right now can’t explain why “The KoF exists” counts for us as false. This is part of a more general difficulty about cancelling too much when the same content is both asserted and presupposed, e.g., “she is a woman.”

I want to consider one final possible application, this one of a very different nature. I have been characterizing subtraction as a way
of cancelling the subtrahend’s content, as contrasted with negating its content. But on a certain view of negation, there is no real contrast here, for negation just is a cancellation device. This view is probably least implausible in the case where the negatum is the very thing just asserted. Here is Strawson in *Introduction to Logical Theory* (p2)

Suppose a man sets out to walk to a certain place; but when he gets half way there, he turns round and comes back again. This may not be pointless. But, from the point of view of change of position it is as if he had never set out. And so a man who contradicts himself may have succeeded in excercising his vocal chords. But from the point of view of imparting information, or communicating facts (or falsehoods), it is as if he had never opened his mouth...the standard function of speech...is frustrated by self-contradiction. Contradiction is like writing something down and erasing it, or putting a line through it. A contradiction cancels itself and leaves nothing.

Strawson in our terms is suggesting that $A$ minus $A$ (= nothing) is the net effect of a speech starting with $A$ and following up with $\sim A$. It hardly needs saying that most people do not think of it that way. Even if Strawson is right that the later $\sim A$ erases the earlier assertion of $A$, why think that $A$ returns the favor, erasing the later assertion of $\sim A$? Berkeley was closer to the mark when he said (*Analyst*):

You may indeed suppose anything possible: But afterwards you may not suppose anything that destroys what you first supposed, or if you do...you may not retain the consequences

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4 I’m indebted in what follows to Priest, “Negation as Cancellation, and Connexive Logic” (Topoi 18, 1999: 141-8).
or any part of the consequences, of the first supposition so destroyed.

Berkeley in our terms is suggesting that the effect of following $A$ with $\neg A$ is not $A$ minus $A$ but $\neg A$ minus $A$, which (taking it that there is no $A$ in $\neg A$ to strip away) is presumably just $\neg A$ again.

But this raises an interesting question. Cancelling a previously made assertion without asserting anything new would seem like a very desirable move in some cases. How in English does one make it, if not by repeating the earlier sentence with “it is not the case that...” stuck on the front? I suggest that at least some of the time we do it by sticking “it might not be the case that...” on the front. To put it more cautiously, if the doctrine of negation as cancellation has any application to English at all, a better target than plain old $it$ is not the case that... is the more complex negative prefix $it$ might not the case that... Let me point out some phenomena that fit ill with the usual view that “might” is used to express ignorance, and better with the hypothesis that it’s some sort of cancellation device. More cautiously let me point out some phenomena that suggest that it signals ignorance, to the extent it does, because it’s a cancellation device, not a cancellation device because it signals ignorance. (There may be other phenomena that point toward the latter view.)

Three main things will be argued: first, that “might” can be used for cancellation purposes; second, cancellation can be a primary effect, not a response to some prior finding of ignorance; and third, the cancellation approach fits well with a popular view of assertion.

So, first, the word “might” turns up of its own accord in what seem like paradigm cancellation contexts. Attempted murder is murder,
except the victim might not die. A lawlike statement is a law, except that it might not be true. One says, “I’ll send the check Thursday morning -- hold on, it might be later in the day,” to cancel the time-of-day aspect and leave in place just the assertion that the check will be sent on that day. Or suppose I am pitching a movie idea to studio executives:

Me: The hero is Raskolnikov, he murders a pawnbroker.  
Exec: That’s no good if we want PG-13.  
Me: Fine, so he might just rough him up a bit.

Second and related, even if we grant that the epistemic theory of “might” has the resources to deal with cases where \( P \) is cancelled because the speaker doesn’t know that \( P \), content can be cancelled for lots of other reasons as well, and “might” remains appropriate even when ignorance is not the issue. What is the point of adding “except the victim might not die”? It is not to admit ignorance of how the story ends; it doesn’t end any way, because there isn’t any particular victim under discussion. When I say the check might be sent later in the day, I may know full well I’m sending the check in the morning when I say it might not be sent until later; the effect is to knock “I’ll send it in the morning” off the list of what I am promising to do, not the list of what I know I’ll do. When I say, “OK, he might just rough him up,” this is to knock Raskolnikov killing the pawnbroker off the list of proposed fictional truths; whether I know Raskolnikov will kill the pawnbroker is not the issue, and it is not even clear there is anything to know.

Here is what a lot of people think about assertion: on the whole and for the most part, one properly asserts that \( P \) only if one knows that \( P \). To the extent that one implicates, in performing a

\[\text{\footnotesize Need not} \]

“Need not” is more common but that the two phrases are somewhat interchangeable just reinforces the point, for “need not” is on its face a way of suspending or cancelling previously imposed requirements.
speech act, that one is performing it properly, the asserter implicates that she knows that \( P \). That it is only an implicature is confirmed by the cancellability test; if you are pressuring me for an answer, I might say, “my boy did not steal those tarts,” adding that although I’m pretty confident I don’t claim to know.

So much is pretty familiar. My point is just that we have the same sort of evidence that one is at most implicating (not claiming) that one does not know in saying that it might be that \( \sim P \). Suppose, for instance, that you are trying to guess which door I’ve hidden the prize behind. It would be perfectly natural for me to say, “It might be door #1 and it might not; I know, of course, it’s just that I’m not proposing right now to tell you.” Working backwards, this suggests that “\( \Diamond \sim P \)” is used, not to perform a speech act in which one claims not to know, but to perform a speech act that is on the whole improper if one does know. Removing \( P \) from the conversational ledger fits the bill. \( P \) made it onto the ledger on the strength of its supposedly being known; so its being struck from the record should go on the whole with its not being known after all.

One very natural objection has to be mentioned even if I have no very clear answer to it. I have suggested that “might” is a cancellation device first and always, a way of conveying epistemic information second and sometimes. But cancellation presupposes there is something there to be cancelled; and an utterance of “it might be that \( \sim P \)” is in order even when no one has asserted \( P \), or anything in the neighborhood.

I agree that this is a big prima facie problem. How to deal with it? One could start by noting that it’s not unheard of for acts that are in the first instance responses to other acts to occur before those

\[ \text{6 I learned of this sort of example from Egan, Weatherson, and Hawthorne.} \]
other acts, and even without them. I’m sure everyone here has been pre-approved for loans they never applied for. A few lucky people have been told “don’t worry, the answer is yes” even before managing to get the marriage proposal out. Or, to move to a less farfetched analogy, consider statements of permission. The teacher says, “It is forbidden to climb that tree, except that you may climb it to recover a kitten or kite.” Here “you may climb the tree to rescue a kitten” strikes something from the list of what was ruled off limits by “It is forbidden to climb that tree.” It might seem at first that all permission statements must be like that; what sense does it make to allow something which had never been forbidden in the first place?

The fact is, though, that permitting the not previously forbidden makes perfect sense. The teacher could have led with, “It is OK to climb that tree to rescue a kitten,” choosing her words carefully because it was not decided yet whether recreational climbing would be forbidden. The point of this “pre-cancellation” of any order not to climb after kittens is clear; it’s to indicate that no orders are forthcoming which would forbid this behavior. It’s a prophylactic if you like against future orders not to climb the tree even after kittens. Of course, there’s nothing to prevent such orders from later being given; but that would mean the administration had changed its mind about what kind of behavior to forbid.

One can imagine a corresponding view about epistemic modals. It would maintain that “it might be that ~P” serves as a prophylactic against future assertions that P – the obvious implicature being that no such assertions would be proper in the present state of information. Of course, there’s nothing to prevent such assertions from later being made, but they will be made only if the epistemic situation changes. No prophylactic is tamper-proof.
Occupies the place where assertion that P might go, thereby indicating that’s not in the offing. Like antibody that binds to sites where the pathogen might go. Or compare to this page intentionally left blank. I’m given the opportunity to assert and rather than just keeping quiet I indicate that I won’t be taking up that opportunity.

Bibliography


