LECTURES ON
REAL OPTIONS

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Use of Option Pricing Methods

- Hard to handle options with NPV methods. Need option pricing methods.

- We will illustrate this with a simple case: The Option to Invest.

Example:

- You are deciding whether to build a plant that would produce widgets. The plant can be built quickly, and will cost $1 million. A careful analysis shows that the present value of the cash flows from the plant, if it were up and running today, is $1.2 million. Should you build the plant?

Answer: Not clear.
You Have an Option to Invest

• Issue is whether you should **exercise** this option.

• If you exercise the option, it will cost you $I = 1$ million. You will receive an asset whose value **today** is $V = 1.2$ million. Of course $V$ might go up or down in the future, as market conditions change.

• Compare to call option on a stock, where $P$ is price of stock and EX is exercise price:

<table>
<thead>
<tr>
<th>Option</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call option on stock:</td>
<td>Max $(P-EX, 0)$</td>
</tr>
<tr>
<td>Option to invest in factory</td>
<td>Max $(V-I, 0)$</td>
</tr>
</tbody>
</table>
Dynamics of Project Value, $V$

$\mu =$ Expected return on $V$. This expected return will be consistent with the project’s (nondiversifiable) risk.

$\delta =$ Payout rate on project. This is the rate of cash payout, as fraction of $V$.

So $\mu = \delta + \text{expected rate of capital gain}$.

1. No Risk.

Then rate of capital gain is:

$$\frac{\Delta V}{V} = (\mu - \delta) \Delta t$$

and $\mu = r_f$, the risk-free interest rate.
2. \( V \) is Risky.

\[
\frac{\Delta V}{V} = (\mu - \delta) \Delta t + \sigma e_t
\]

where \( e_t \) is random, zero-mean. So \( V \) follows a random walk, like the price of a stock.

- If all risk is diversifiable, \( \mu = r_f \).
- If there is nondiversifiable risk, \( \mu > r_f \).

- Formally, write process for \( V \) as:

\[
\frac{dV}{V} = (\mu - \delta)dt + \sigma dz
\]

where \( dz = e_t \sqrt{dt} \) is the increment of a Wiener process, and \( e_t \) normally distributed, with \((dz)^2 = dt\).

- So \( V \) follows a geometric Brownian motion (GBM).
- Note \( (dV)^2 = \sigma^2 V^2 dt \).
Mathematical Background

• **Wiener Process:** If $z(t)$ is a Wiener process, any change in $z$, $\Delta z$, over time interval $\Delta t$, satisfies

1. The relationship between $\Delta z$ and $\Delta t$ is given by:

   \[ \Delta z = \epsilon_t \sqrt{\Delta t}, \]

   where $\epsilon_t$ is a normally distributed random variable with zero mean and a standard deviation of 1.

2. The random variable $\epsilon_t$ is serially uncorrelated, i.e., $\mathbb{E}[\epsilon_t \epsilon_s] = 0$ for $t \neq s$. Thus values of $\Delta z$ for any two different time intervals are independent.

• What do these conditions imply for change in $z$ over an interval of time $T$?

   - Break interval up into $n$ units of length $\Delta t$ each, with $n = T/\Delta t$. Then the change in $z$ over this interval is

   \[ z(s + T) - z(s) = \sum_{i=1}^{n} \epsilon_i \sqrt{\Delta t} \quad (1) \]
The $\epsilon_i$’s are independent of each other. By the Central Limit Theorem, the change $z(s+T) - z(s)$ is normally distributed with mean 0 and variance $n \Delta t = T$.

- This result, which follows from the fact that $\Delta z$ depends on $\sqrt{\Delta t}$ and not on $\Delta t$, is particularly important; the variance of the change in a Wiener process grows linearly with the time horizon.

- By letting $\Delta t$ become infinitesimally small, we can represent the increment of a Wiener process, $dz$, as:

$$dz = \epsilon_t \sqrt{dt}$$

(2)

- Since $\epsilon_t$ has zero mean and unit standard deviation, $\mathcal{E}(dz) = 0$, and $\mathcal{V}[dz] = \mathcal{E}[(dz)^2] = dt$.

- Note that a Wiener process has no time derivative in a conventional sense; $\Delta z/\Delta t = \epsilon_t (\Delta t)^{-1/2}$, which becomes infinite as $\Delta t$ approaches zero.
• **Generalization: Brownian Motion with Drift.**

\[ dx = \alpha \, dt + \sigma \, dz, \]  

where \( dz \) is the increment of a Wiener process, \( \alpha \) is the drift parameter, and \( \sigma \) the variance parameter.

• Over any time interval \( \Delta t \), \( \Delta x \) is normally distributed, and has expected value \( \mathcal{E}(\Delta x) = \alpha \Delta t \) and variance \( \mathcal{V}(\Delta x) = \sigma^2 \Delta t \).

• Figure 3.1 shows three sample paths of eq. (3), with \( \alpha = 0.2 \) per year, and \( \sigma = 1.0 \) per year. Each sample path was generated by taking a \( \Delta t \) of one month, and then calculating a trajectory for \( x(t) \) using

\[ x_t = x_{t-1} + .01667 + .2887 \epsilon_t, \]  

with \( x_{1950} = 0 \). In eq. (4), at each time \( t \), \( \epsilon_t \) is drawn from a normal distribution with zero mean and unit standard deviation. (Note: \( \alpha \) and \( \sigma \) are in monthly terms. Trend of .2 per year implies 0.0167 per month; S.D. of 1.0 per year implies variance of 1.0 per year, hence variance of \( \frac{1}{12} = .0833 \) per month, so monthly S.D. is \( \sqrt{0.0833} = 0.2887 \).) Also shown is a trend line, i.e., eq. (4) with \( \epsilon_t = 0 \).
Fig. 3.2 shows a forecast of this process. A sample path was generated from 1950 to the end of 1974, again using eq. (4), and then forecasts of $x(t)$ were constructed for 1975 to 2000. The forecast of $x$ for $T$ months beyond Dec. 1974 is

$$\hat{x}_{1974+T} = x_{1974} + .01667T.$$ 

The graph also shows a 66 percent forecast confidence interval, i.e., the forecasted $x(t)$ plus or minus one standard deviation. Recall that variance grows linearly with the time horizon, so the standard deviation grows as the square root of the time horizon. Hence the 66 percent confidence interval is

$$x_{1974} + .01667T \pm .2887 \sqrt{T}.$$ 

One can similarly construct 90 or 95 percent confidence intervals.
Figure 3.1: Sample Paths of Brownian Motion with Drift

Figure 3.2: Optimal Forecast of Brownian Motion with Drift
Mathematical Background (Cont’d)

• Generalized Brownian Motion: Ito Process.

$$dx = a(x, t)dt + b(x, t)dz$$

− Mean of $dx$: $\mathcal{E}(dz) = 0$ so $\mathcal{E}(dx) = \alpha(x, t)dt$.

− Variance of $dx$: $\mathcal{V}(dx) = \mathcal{E}[dx^2] - (\mathcal{E}[dx]^2)$, which has terms in $dt$, $(dt)^2$, and $(dt)(dz)$, which is of order $(dt)^{3/2}$. For $dt$ infinitesimally small, terms in $(dt)^2$ and $(dt)^{3/2}$ can be ignored, so

$$\mathcal{V}[dx] = b^2(x, t)dt.$$

• Geometric Brownian Motion (GBM).

$a(x, t) = \alpha x$, and $b(x, t) = \sigma x$, so

$$dx = \alpha x dt + \sigma x dz.$$  \hspace{1cm} (5)

− Percentage changes in $x$, $\Delta x / x$, are normally distributed, so absolute changes in $x$, $\Delta x$, are log-normally distributed.

− If $x(0) = x_0$, the expected value of $x(t)$ is

$$\mathcal{E}[x(t)] = x_0 e^{\alpha t},$$
and the variance of $x(t)$ is

$$\mathcal{V}[x(t)] = x_0^2 e^{2\alpha t} (e^{\sigma^2 t} - 1).$$

Can calculate the expected present discounted value of $x(t)$ over some period of time. For example,

$$\mathcal{E} \left[ \int_0^\infty x(t) e^{-rt} dt \right] = \int_0^\infty x_0 e^{-(r-\alpha)t} dt = x_0/(r - \alpha). \quad (6)$$
Mathematical Background (Cont’d)

• Fig. 3.4 shows three sample paths of eq. (5), with $\alpha = .09$ and $\sigma = .2$. (These numbers approximately equal annual expected growth rate and S.D. of the NYSE Index in real terms.) Sample paths were generated by taking a $\Delta t$ of one month, and using

$$x_t = 1.0075 x_{t-1} + .0577 x_{t-1} \epsilon_t, \quad (7)$$

with $x_{1950} = 100$. Also shown is the trend line.

• Note that in one of these sample paths the “stock market” outperformed its expected rate of growth, but in the other two it underperformed.

• Figure 3.5 shows a forecast of this process. The forecasted value of $x$ is given by

$$\hat{x}_{1974+T} = (1.0075)^T x_{1974},$$

where $T$ is in months. Also shown is a 66-percent confidence interval. Since the S.D. of percentage changes in $x$ grows with the square root of the time horizon, bounds of this confidence interval are

$$(1.0075)^T (1.0577)^{\sqrt{T}} x_{1974} \quad \text{and} \quad (1.0075)^T (1.0577)^{-\sqrt{T}} x_{1974}.$$
Figure 3.4: Sample Paths of Geometric Brownian Motion

Figure 3.5: Optimal Forecast of Geometric Brownian Motion
Mathematical Background (Cont’d)

• **Mean-Reverting Processes:** Simplest is

\[ dx = \eta (\bar{x} - x) \, dt + \sigma \, dz. \]  

(8)

Here, \( \eta \) is speed of reversion, and \( \bar{x} \) is “normal” level of \( x \).

• If \( x = x_0 \), its expected value at future time \( t \) is

\[ \mathcal{E}[x_t] = \bar{x} + (x_0 - \bar{x}) \, e^{-\eta t}. \]  

(9)

Also, the variance of \( (x_t - \bar{x}) \) is

\[ \mathcal{V}[x_t - \bar{x}] = \frac{\sigma^2}{2\eta} \left(1 - e^{-2\eta t}\right). \]  

(10)

• Note that \( \mathcal{E}[x_t] \to \bar{x} \) as \( t \) becomes large, and variance \( \to \sigma^2/2\eta \). Also, as \( \eta \to \infty \), \( \mathcal{V}[x_t] \to 0 \), so \( x \) can never deviate from \( \bar{x} \). As \( \eta \to 0 \), \( x \) becomes a simple Brownian motion, and \( \mathcal{V}[x_t] \to \sigma^2 t \).

• Fig. 3.6 shows four sample paths of equation (8) for different values of \( \eta \). In each case, \( \sigma = .05 \) in *monthly* terms, \( \bar{x} = 1 \), and \( x(t) \) begins at \( x_0 = 1 \).
Fig. 3.7 shows an optimal forecast for $\eta = .02$, along with a 66-percent confidence interval. Note that after four or five years, the variance of the forecast converges to $\sigma^2/2\eta = .0025/.04 = .065$, so the 66 percent confidence interval ($\pm 1$ S.D.) converges to the forecast $\pm .25$.

Can generalize eq. (8). For example, one might expect $x(t)$ to revert to $\bar{x}$ but the variance rate to grow with $x$. Then one could use

$$dx = \eta (\bar{x} - x) \, dt + \sigma x \, dz.$$ \hfill (11)

Or, proportional changes in a variable might be modelled as mean-reverting:

$$dx = \eta x (\bar{x} - x) \, dt + \sigma x \, dz.$$ \hfill (12)
Figure 3.6: Sample Paths of Mean-Reverting Process:

\[ dx = \eta(\bar{x} - x)dt + \sigma dz \]

Figure 3.7: Optimal Forecast of Mean-Reverting Process
• **Ito’s Lemma**: Suppose \( x(t) \) is an Ito process, i.e.,
\[
dx = a(x, t)dt + b(x, t)dz , \tag{13}
\]

How do we take derivatives of \( F(x, t) \)?

− Usual rule of calculus:
\[
dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial t} \, dt.
\]

− But suppose we include higher order terms:
\[
dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial t} \, dt + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dx)^2 + \frac{1}{6} \frac{\partial^3 F}{\partial x^3} (dx)^3 + \ldots \tag{14}
\]

In ordinary calculus, these higher order terms all vanish in the limit. Is that the case here? First, substitute eq. (13) for \( dx \) to determine \((dx)^2\):
\[
(dx)^2 = a^2(x, t) \, (dt)^2 + 2 a(x, t) b(x, t) \, (dt)^{3/2} + b^2(x, t) \, dt.
\]

Terms in \((dt)^{3/2}\) and \((dt)^2\) go to zero faster than \(dt\) as it becomes infinitesimally small, so we can ignore these terms and write
\[
(dx)^2 = b^2(x, t) \, dt.
\]
Every term in the expansion of \((dx)^3\) will include \(dt\) to a power greater than 1, and so will go to zero faster than \(dt\) in the limit. Likewise for \((dx)^4\), etc. Hence the differential \(dF\) is

\[
dF = \frac{\partial F}{\partial t} \, dt + \frac{\partial F}{\partial x} \, dx + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \, (dx)^2. \tag{15}
\]

Can write this in expanded form by substituting equation (13) for \(dx\):

\[
dF = \left[ \frac{\partial F}{\partial t} + a(x, t) \frac{\partial F}{\partial x} + \frac{1}{2} b^2(x, t) \frac{\partial^2 F}{\partial x^2} \right] \, dt \\
+ b(x, t) \frac{\partial F}{\partial x} \, dz. \tag{16}
\]
Example: Geometric Brownian Motion. Let’s return to the GBM of eq. (5). Use Ito’s Lemma to find the process given by $F(x) = \log x$.

Since $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = 1/x$, and $\frac{\partial^2 F}{\partial x^2} = -1/x^2$, we have from (15):

$$dF = \frac{1}{x} \, dx - \frac{1}{2x^2} \, (dx)^2$$
$$= \alpha \, dt + \sigma \, dz - \frac{1}{2} \sigma^2 \, dt$$
$$= (\alpha - \frac{1}{2} \sigma^2) \, dt + \sigma \, dz. \quad (17)$$

Hence over any interval $T$, the change in $\log x$ is normally distributed with mean $(\alpha - \frac{1}{2} \sigma^2)T$ and variance $\sigma^2 T$.

Why is the drift rate of $F(x) = \log x$ less than $\alpha$? Because $\log x$ is a concave function of $x$, so with $x$ uncertain, expected value of $\log x$ changes by less than the log of the expected value of $x$. Uncertainty over $x$ is greater the longer the time horizon, so the expected value of $\log x$ is reduced by an amount that increases with time; hence drift rate is reduced.
Investment Problem

• Invest $I$, get $V$, with $V$ a GBM, i.e.,

$$dV = \alpha V \, dt + \sigma V \, dz$$  \hspace{1cm} (18)

Payout rate is $\delta$ and expected rate of return is $\mu$, so $\alpha = \mu - \delta$.

• Want value of the investment opportunity (i.e., the option to invest), $F(V)$, and decision rule $V^*$.

• Solution by Dynamic Programming: $F(V)$ must satisfy Bellman equation:

$$\rho F \, dt = \mathcal{E}(dF).$$  \hspace{1cm} (19)

- This just says that over interval $dt$, total expected return on the investment opportunity, $\rho F \, dt$, is equal to expected rate of capital appreciation.

- Expand $dF$ using Ito’s Lemma:

$$dF = F'(V) \, dV + \frac{1}{2} F''(V) \, (dV)^2.$$  

- Substituting eq. (18) for $dV$ and noting that $\mathcal{E}(dz) = 0$ gives

$$\mathcal{E}[dF] = \alpha V F'(V) \, dt + \frac{1}{2} \sigma^2 V^2 F''(V) \, dt.$$
Hence the Bellman equation becomes (after dividing through by $dt$):

$$\frac{1}{2} \sigma^2 V^2 F''(V) + \alpha V F'(V) - \rho F = 0. \quad (20)$$

Also, $F(V)$ must satisfy boundary conditions:

$$F(0) = 0 \quad (21)$$

$$F(V^*) = V^* - I \quad (22)$$

$$F'(V^*) = 1 \quad (23)$$

This is easy to solve, but first, let’s do this again using option pricing approach. Then we avoid problem of choosing $\rho$ and $\alpha$. 
Investment Problem (Cont’d)

- Solution by Contingent Claims Analysis.
- Create a risk-free portfolio: Hold option to invest, worth \( F(V) \). Short \( n = dF/dV \) units of project.

\[
\text{Value of portfolio } = \Phi = F - F'(V)V
\]

- Short position requires payment of \( \delta VF'(V) \) dollars per period. So total return on portfolio is:

\[
dF - F'(V)dV - \delta VF'(V)dt
\]

Since \( dF = F'(V)dV + \frac{1}{2}F''(V)(dV)^2 \), total return is:

\[
\frac{1}{2}F''(V)(dV)^2 - \delta VF'(V)dt
\]

\((dV)^2 = \sigma^2V^2dt\), so total return becomes:

\[
\frac{1}{2}\sigma^2V^2F''(V)dt - \delta VF'(V)dt
\]

- This return is risk-free. Hence to avoid arbitrage, it must equal \( r\Phi dt = r[F - F'(V)V]dt \):

\[
\frac{1}{2}\sigma^2V^2F''(V)dt - \delta VF'(V)dt = r[F - F'(V)V]dt
\]
Implies the following differential equation that $F(V)$ must satisfy:

$$\frac{1}{2} \sigma^2 V^2 F''(V) + (r - \delta) VF'(V) - r F = 0$$

Boundary conditions:

$$F(0) = 0 \quad (24)$$
$$F(V^*) = V^* - I \quad (25)$$
$$F'(V^*) = 1 \quad (26)$$

Solution

$$F(V) = AV^{\beta}, \quad \text{for } V \leq V^*$$
$$= V - I, \quad \text{for } V > V^*$$

where $\beta = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{(\frac{r - \delta}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$, $V^* = \frac{\beta I}{\beta - 1}$, $A = \frac{V^* - I}{(V^*)^\beta}$
Characteristics of Solution

• Higher $\sigma$: Makes $F$ larger, and also makes $V^*$ larger. Now the firm’s options to invest (patents, land, etc.) are worth more, but the firm should do less investing. Opportunity cost of investing is higher.

• Higher $\delta$: Higher cash payout rate increases opportunity cost of keeping the option alive. Makes $F$ smaller, $V^*$ smaller. Now the firm’s options to invest are worth less, but the firm should do more investing.

• Higher $r$: Makes $F$ larger and $V^*$ larger. Option value increases because present value of exercise price $I$ decreases (in event of exercise). This increases opportunity cost of “killing” the option, so $V^*$ rises.

• Higher $I$: Makes $F$ smaller and $V^*$ larger. Note that as $I \to 0$, $F \to V$, and $V^* \to 0$.

• Question: Suppose $\sigma$ increases. We know this makes $V^*$ larger. What happens to $\mathcal{E}(T^*)$, the expected time until $V$ reaches $V^*$ and investment occurs?
Figure 5.3: Value of Investment Opportunity

Figure 5.4: Critical Value $V^*$ as a Function of $\sigma$
How Important is Option Value?

- Take a factory which costs $1 million (= I) to build. Suppose riskless rate is 7% (nominal). Critical value $V^*$ is shown below for different values of $\sigma$ and $\delta$.

<table>
<thead>
<tr>
<th>Annual Standard Deviation of Project Value ($\sigma$)</th>
<th>Payout Rate, $\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>2%</td>
</tr>
<tr>
<td>10%</td>
<td>3.84</td>
</tr>
<tr>
<td>20%</td>
<td>4.77</td>
</tr>
<tr>
<td>40%</td>
<td>8.06</td>
</tr>
</tbody>
</table>

- For average firm on NYSE, $\sigma = 20\%$.

- Observe that for $\sigma = 20\%$ or 40\%, $V^*$ is much larger than $I$. So how important is option value? Very.
How Important is Option Value? (Cont’d)

- How valuable is option to invest, even if not exercised? Suppose $V = I = $1 million. Option value, $F$, is shown below.

<table>
<thead>
<tr>
<th>Annual Standard Deviation of Project Value ($\sigma$)</th>
<th>2%</th>
<th>5%</th>
<th>10%</th>
<th>15%</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>.46</td>
<td>.18</td>
<td>.05</td>
<td>.02</td>
</tr>
<tr>
<td>20%</td>
<td>.52</td>
<td>.27</td>
<td>.12</td>
<td>.07</td>
</tr>
<tr>
<td>40%</td>
<td>.65</td>
<td>.44</td>
<td>.28</td>
<td>.20</td>
</tr>
</tbody>
</table>
Mean-Reverting Process

• Suppose \( V \) follows the mean-reverting process:

\[
dV = \eta (\bar{V} - V) V \, dt + \sigma V \, dz ,
\]

(27)

• To find the optimal investment rule, use contingent claims analysis.

  – Let \( \mu \) = risk-adjusted discount rate for project.

  – Expected rate of growth of \( V \) is not constant, but a function of \( V \). Hence the “shortfall,” \( \delta = \mu - (1/dt)\mathcal{E}(dV)/V \), is a function of \( V \):

\[
\delta(V) = \mu - \eta (\bar{V} - V) .
\]

(28)

  – The differential equation for \( F(V) \) is now

\[
\frac{1}{2} \sigma^2 V^2 F''(V) + [r - \mu + \eta (\bar{V} - V)] V F'(V) - r F = 0 .
\]

(29)

  – Also, \( F(V) \) must satisfy the same boundary conditions (21) – (23) as before, and for the same reasons.

• Solution is a little more complicated. Define a new
function $h(V)$ by

$$F(V) = AV^\theta h(V), \quad (30)$$

Substituting this into eq. (29) and rearranging gives the following equation:

$$V^\theta h(V) \left[ \frac{1}{2} \sigma^2 \theta(\theta - 1) + (r - \mu + \eta \bar{V}) \theta - r \right]$$

$$+ V^{\theta+1} \left[ \frac{1}{2} \sigma^2 V h''(V) + (\sigma^2 \theta + r - \mu + \eta \bar{V} - \eta V) h'(V) - \eta \theta h(V) \right] = 0. \quad (31)$$

Eq. (31) must hold for any value of $V$, so the bracketed terms in both the first and second lines must equal zero. First choose $\theta$ to set the bracketed terms in the first line equal to zero:

$$\frac{1}{2} \sigma^2 \theta(\theta - 1) + (r - \mu + \eta \bar{V}) \theta - r = 0.$$

To satisfy the boundary condition that $F(0) = 0$, we use the positive solution:

$$\theta = \frac{1}{2} + \left( \mu - r - \eta \bar{V} / \sigma^2 \right) + \sqrt{\left( \mu - r - \eta \bar{V} / \sigma^2 \right)^2 + 2r / \sigma^2}. \quad (32)$$

From the second line of eq. (31),

$$\frac{1}{2} \sigma^2 V h''(V) + (\sigma^2 \theta + r - \mu + \eta \bar{V} - \eta V) h'(V) - \eta \theta h(V) = 0. \quad (33)$$
Make the substitution $x = 2\eta V/\sigma^2$, to transform eq. (33) into a standard form. Let $h(V) = g(x)$, so that $h'(V) = (2\eta/\sigma^2) g'(x)$ and $h''(V) = (2\eta/\sigma^2)^2 g''(x)$. Then (33) becomes

$$x g''(x) + (b - x) g'(x) - \theta g(x) = 0 ,$$  \hspace{1cm} (34)

where

$$b = 2\theta + 2 (r - \mu + \eta \bar{V})/\sigma^2 .$$

Eq. (34) is Kummer’s Equation. Its solution is the confluent hypergeometric function $H(x; \theta, b(\theta))$, which has the following series representation:

$$H(x; \theta, b) = 1 + \frac{\theta}{b} x + \frac{\theta(\theta + 1)}{b(b + 1)} \frac{x^2}{2!} + \frac{\theta(\theta + 1)(\theta + 2)}{b(b + 1)(b + 2)} \frac{x^3}{3!} + \ldots .$$  \hspace{1cm} (35)

We have verified that the solution to equation (29) is indeed of the form of equation (30). Solution is

$$F(V) = A V^\theta H(\frac{2\eta}{\sigma^2} V; \theta, b) ,$$  \hspace{1cm} (36)

where $A$ is a constant yet to be determined.
We can find $A$, as well as the critical value $V^*$, from the remaining two boundary conditions, that is, $F(V^*) = V^* - I$ and $F_V(V^*) = 1$. Because the confluent hypergeometric function is an infinite series, $A$ and $V^*$ must be found numerically.

- Look at several numerical solutions. Set $I = 1$, $r = .04$, $\mu = .08$, and $\sigma = .2$. We will vary $\eta$ and $\bar{V}$. 

Figure 5.11: Mean Reversion — $F(V)$ for $\eta = 0.05$ and $\tilde{V} = 0.5, 1.0, \text{ and } 1.5$

Figure 5.12: Mean Reversion — $F(V)$ for $\eta = 0.1$ and $\tilde{V} = 0.5, 1.0, \text{ and } 1.5$
Figure 5.13: Mean Reversion — $F(V)$ for $\eta = 0.05$ and $\bar{V} = 0.5$, 1.0, and 1.5

Figure 5.14: Critical Value $V^*$ as a Function of $\eta$ for $\mu = .08$ and $\bar{V} = 0.5$, 1.0, and 1.5
• An undeveloped oil reserve is like a call option. It gives the owner the right to acquire a developed reserve by paying the development cost.

• When to develop the reserve? Same as deciding when to exercise a call option.

• The higher the uncertainty over future oil prices, the more valuable is the undeveloped reserve, and the longer we should wait to develop it.
<table>
<thead>
<tr>
<th>Stock Call Option</th>
<th>Undeveloped Reserve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current stock price</td>
<td>Current value of developed reserve</td>
</tr>
<tr>
<td>Variance of rate of return on the stock</td>
<td>Variance of rate of change of the value of a developed resource</td>
</tr>
<tr>
<td>Exercise price</td>
<td>Development Cost</td>
</tr>
<tr>
<td>Time to expiration</td>
<td>Relinquishment requirement</td>
</tr>
<tr>
<td>Riskless rate of interest</td>
<td>Riskless rate of interest</td>
</tr>
<tr>
<td>Dividend</td>
<td>Net production revenue less depletion</td>
</tr>
</tbody>
</table>
Valuing Undeveloped Oil Reserves and Making Development Decisions

1. Value of Developed Reserve

\[ B = \text{number of barrels of oil} \]
\[ V = \text{value per barrel} \]
\[ R = \text{rate of return to owner of reserve} \]
\[ P = \text{price of a barrel of oil} \]
\[ \Pi = \text{After-tax profit from producing and selling a barrel of oil} \]
\[ \mu = \text{risk-adjusted expected rate of return on oil} \]
\[ \omega = \text{fraction produced each year} = .10 \]

- Production cost \( \approx .3P \), tax rate \( \approx .34 \),

so \( \Pi = (1 - .34)(.7)P = .45P \)

\[ V = \int_0^\infty \omega \Pi e^{-(r+\omega)t} dt = \frac{\omega (.45P)}{r + \omega} = \frac{.045P}{.14} = .32P \]

- Depletion of reserve: \( dB = -\omega B dt \)
• Return dynamics:

\[ Rdt = \omega B\Pi dt + d(BV) \]

\[ = \omega B\Pi dt + BdV - \omega BV dt \]

• Return is partly random:

\[ \frac{Rdt}{BV} = \mu dt + \sigma dz \]

where \( dz = \epsilon \sqrt{dt} \)

• \( \mu BV dt + \sigma BV dz = \omega B\Pi dt + BdV - \omega BV dt \)
so \( dV = \mu V dt - \omega (\Pi - V) dt + \sigma V dz \)

• Therefore: \( dV = (\mu - \delta)V dt + \sigma V dz \) where \( \delta \) is payout rate net of depletion:

\[ \delta = \omega (\Pi - V)/V \approx .04 \]
2. Value of Undeveloped Reserve

Let $F(V, t) = \text{value of 1-barrel unit of undeveloped reserve}$.

- By setting up risk-free portfolio, etc., can show that $F$ must satisfy:

\[
\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} + (r - \delta)V \frac{\partial F}{\partial V} - rF = -\frac{\partial F}{\partial t}
\]

- Boundary conditions:

$F(0, t) = 0$

$F(V, T) = \max[V_T - D, 0]$  

$F(V^*, t) = V^* - D$

$\frac{\partial F(V^*, t)}{\partial V} = 1$

where $D = \text{per barrel cost of development}$  

$V^* = \text{critical value that triggers development}$  

$T = \text{time to expiration}$
Valuation Example

• developed reserve expected to yield 100 million barrels of oil;

• the present value of the development cost is $11.79 per barrel;

• the development lag is 3 years;

• relinquishment is after a period of 10 years;

• standard deviation of value of developed reserves is 14.2 percent;

• the payout ratio (net production revenues/value of reserves) is 4.1 percent; and

• the value of the developed reserve today is $12 per barrel.

**Step 1:** Calculate the present value of a developed reserve \( V' \): 
\[
V' = e^{-0.041 \times 3} \times 12 = 10.61 \text{ per barrel.}
\]

**Step 2:** Calculate the ratio of reserve value to development cost, \( C \).
\[
C = \frac{V'}{D} = \frac{10.61}{11.79} = .90.
\]
Valuation Example (Cont’d)

Step 3: Calculate the value of undeveloped reserves:

Value = (Option Value per $1 Development Cost (from Table)) × (Total Development Cost) = (0.05245) × ($1179.0 million) = $61.84 million.

- Hence, although reserve cannot be profitably developed under current conditions, the right to develop it in the future is worth more than $60 million.
Optimal Timing of Exercise

Critical Value for Development of Oil Reserve
(Shows $V^* / D$ for $\delta = 0.04$ and $r = 0.0125$, where $D$ is development cost)
### Option Values per $1 of Development Cost

<table>
<thead>
<tr>
<th>(V/D)</th>
<th>(\sigma_r = 0.142)</th>
<th>(\sigma_r = 0.25)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(T = 5)</td>
<td>(T = 10)</td>
</tr>
<tr>
<td>0.80</td>
<td>0.01810</td>
<td>0.02812</td>
</tr>
<tr>
<td>0.85</td>
<td>0.02761</td>
<td>0.03894</td>
</tr>
<tr>
<td>0.90</td>
<td>0.04024</td>
<td>0.05245</td>
</tr>
<tr>
<td>0.95</td>
<td>0.05643</td>
<td>0.06899</td>
</tr>
<tr>
<td>1.00</td>
<td>0.07661</td>
<td>0.08890</td>
</tr>
<tr>
<td>1.05</td>
<td>0.10116</td>
<td>0.11253</td>
</tr>
<tr>
<td>1.10</td>
<td>0.13042</td>
<td>0.14025</td>
</tr>
<tr>
<td>1.15</td>
<td>0.16472</td>
<td>0.17242</td>
</tr>
</tbody>
</table>


Note: Because option values are homogeneous in the development cost, total option value is the entry in the table times the total development cost.
## Undeveloped Reserve Values

(in $ Millions)

<table>
<thead>
<tr>
<th>V' / D</th>
<th>$σ = 14.2%$</th>
<th>$σ = 25%$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T=5</td>
<td>T=10</td>
</tr>
<tr>
<td>0.70</td>
<td>7.72</td>
<td>15.59</td>
</tr>
<tr>
<td>0.80</td>
<td>21.34</td>
<td>33.15</td>
</tr>
<tr>
<td>0.90</td>
<td>47.44</td>
<td>61.84</td>
</tr>
<tr>
<td>1.00</td>
<td>90.32</td>
<td>104.81</td>
</tr>
<tr>
<td>1.10</td>
<td>153.77</td>
<td>165.35</td>
</tr>
</tbody>
</table>

Note: This table uses a payout ratio of 4.1 percent and 100 Million Bbls of Oil.
Is Geometric Brownian Motion the Right Model of Price?

Log Price of Crude Oil and Quadratic Trend Lines
Mean Reversion in Oil Price

Consider the following mean-reverting process for the value of a developed oil reserve:

\[ dV = \eta (V - \bar{V}) V \, dt + \sigma V \, dz. \quad (37) \]

Then the partial differential equation for value of undeveloped reserve, \( F(V, t) \), is

\[
\frac{1}{2} \sigma^2_v V^2 F_{VV} + [r - \mu + \eta(V - \bar{V})] V \, F_V - r \, F = -F_t
\]

(38)

If the time until relinquishment is long enough (more than five years), we can ignore the time dependence of \( F(V, t) \), so that the term \(-F_t\) disappears.

- As we saw, solution can be written using the confluent hypergeometric function, which has a series representation.

- Can use this solution, with different values for \( \eta \) and \( \bar{V} \), to determine the extent to which mean reversion matters.
• Wey (1993) has shown, using a 100-year series for the real price of crude oil, that a reasonable estimate for $\eta$ is about 0.3, and that using this value (and a value of $\sigma_v$ of .20), the extent to which mean reversion matters depends on the value to which $V$ reverts, $\overline{V}$, relative to the development cost, $D$.

- If $\overline{V}$ is much larger than $D$, accounting for mean reversion gives a larger value for the undeveloped reserve when $V < D$ because $V$ is expected to rise over time. Wey shows that if $\overline{V}$ is about twice as large as $D$, ignoring mean reversion can lead one to undervalue the reserve by 40 percent or more.

- On the other hand, if $\overline{V}$ is about as large as $D$, ignoring mean reversion will matter very little.
Sequential Investment

• Many investment decisions are made sequentially, and in a particular order.

  – Oil production capacity: Find reserves, then develop them.
  – New line of aircraft: Engineering, prototype production, testing, final tooling.
  – Drug development: Find new molecule, Phase I testing, Phase II, Phase III, construct production facility, marketing.
  – Any investment that can be halted midway and temporarily or permanently abandoned.

• Like a compound option; each stage completed (or dollar invested) gives the firm an option to complete the next stage (or invest the next dollar).
• **Example:** two-stage investment in new oil production capacity.

  – First, obtain reserves, through exploration or purchase, at cost \( I_1 \).
  
  – Second, Build development wells, at cost \( I_2 \).
  
  – Begin with option, worth \( F_1(P) \), to invest in reserves. Investing gives the firm another option, worth \( F_2(P) \), to invest in development wells.

  – Making this second investment yields production capacity, worth \( V(P) \).

• Work backwards to find the optimal investment rules.
Investment Rule for a Two-Stage Project

- Project, once completed, produces one unit of output per period at an operating cost $C$.
  - Output can be sold at price $P$, which follows a GBM:
    \[
    dP = \alpha P \, dt + \sigma P \, dz. \tag{39}
    \]
  - Production can be temporarily suspended when $P$ falls below $C$, and resumed when $P$ rises above $C$, so profit flow is $\pi(P) = \max [P - C, 0]$.

- Investing in first stage requires sunk cost $I_1$, and second stage requires sunk cost $I_2$.

- Solve the investment problem by working backwards:
  - First, find value of completed project $V(P)$.
  - Next, find value of option to invest in second stage, $F_2(P)$, and critical price $P_2^*$ for investing.
  - Then, find value of option to invest in first stage, $F_1(P)$, and critical price $P_1^*$.
• Value of the Project. $V(P)$ must satisfy

$$\frac{1}{2} \sigma^2 P^2 V''(P) + (r - \delta) PV'(P) - r V(P) + \pi(P) = 0, \quad (40)$$

subject to $V(0) = 0$, and continuity of $V(P)$ and $V_P(P)$ at $P = C$. Solution is:

$$V(P) = \begin{cases} 
A_1 P^{\beta_1} & \text{if } P < C, \\
B_2 P^{\beta_2} + P/\delta - C/r & \text{if } P > C.
\end{cases} \quad (41)$$

where

$$\beta_1 = \frac{1}{2} - (r - \delta)/\sigma^2 + \sqrt{[(r - \delta)/\sigma^2 - \frac{1}{2}]^2 + 2r/\sigma^2} > 1,$$

$$\beta_2 = \frac{1}{2} - (r - \delta)/\sigma^2 - \sqrt{[(r - \delta)/\sigma^2 - \frac{1}{2}]^2 + 2r/\sigma^2} < 0.$$

Constants $A_1$ and $B_2$ found from continuity of $V(P)$ and $V'(P)$ at $P = C$:

$$A_1 = \frac{C^{1-\beta_1}}{\beta_1 - \beta_2} \left( \frac{\beta_2}{r} - \frac{\beta_2 - 1}{\delta} \right) \quad (42)$$

$$B_2 = \frac{C^{1-\beta_2}}{\beta_1 - \beta_2} \left( \frac{\beta_1}{r} - \frac{\beta_1 - 1}{\delta} \right) \quad (43)$$
• Second-Stage Investment. Find value of option to invest in second stage, \( F_2(P) \), and critical price \( P_2^* \).

- Value of option must satisfy
  \[
  \frac{1}{2} \sigma^2 P^2 F''_2(P) + (r - \delta) P F'_2(P) - r F(P) = 0
  \]
  subject to the boundary conditions
  \[
  F_2(0) = 0 \quad (45)
  \]
  \[
  F_2(P_2^*) = V(P_2^*) - I_2 \quad (46)
  \]
  \[
  F'_2(P_2^*) = V'(P_2^*) \quad (47)
  \]

- Guess and then confirm that \( P_2^* > C \), so use solution for \( V(P) \) in eq. (41) for \( P > C \), in conditions (46) and (47). From condition (45):
  \[
  F_2(P) = D_2 \, P^{\beta_1} \, .
  \]
  From boundary conditions (46) and (47),
  \[
  D_2 = \frac{\beta_2 B_2}{\beta_1} (P_2^*)^{(\beta_2 - \beta_1)} + \frac{1}{\delta \beta_1} (P_2^*)^{(1 - \beta_1)},
  \]
  and \( P_2^* \) is the solution to
  \[
  (\beta_1 - \beta_2) B_2 (P_2^*)^{\beta_2} + (\beta_1 - 1) \frac{P_2^*}{\delta} - \beta_1 \left( \frac{C}{r} + I_2 \right) = 0.
  \]
  Eq. (50) must be solved numerically for \( P_2^* \).
Solution given by eq. (48) applies for $P < P_2^*$. When $P \geq P_2^*$ the firm exercises option to invest, and $F_2(P) = V(P) - I_2$:

$$F_2(P) = \begin{cases} D_2 \ P^\beta_1 & \text{for } P < P_2^* \\ V(P) - I_2 & \text{for } P \geq P_2^* \end{cases}$$  \hspace{1cm} (51)

- **First-Stage Investment.** Given $F_2(P)$ and $P_2^*$, can back up to first stage and find value of option to invest, $F_1(P)$, and critical price $P_1^*$.

  - $F_1(P)$ also satisfies eq. (44), but now subject to
  
  $$F_1(0) = 0$$  \hspace{1cm} (52)
  $$F_1(P_1^*) = F_2(P_1^*) - I_1$$  \hspace{1cm} (53)
  $$F_1'(P_1^*) = F_2'(P_1^*)$$  \hspace{1cm} (54)

  - Solution has the usual form:

$$F_1(P) = D_1 \ P^\beta_1.$$ \hspace{1cm} (55)

Use (53) and (54) to find $D_1$ and critical price $P_1^*$. Because $P_1^* > P_2^*$, $F_2(P_1^*) = V(P_1^*) - I_2$. 

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• Since $P_1^* > P_2^*$, and since the investment can be completed instantaneously (before $P$ can change), we know that once $P$ reaches $P_1^*$ and the firm invests, it will complete both stages of the project. This result seems anti-climactic, so why bother to solve this two-stage problem? Why not simply combine the two stages?

  – First, real-world investing takes time, so firms often do complete early stages and then wait before proceeding with later stages.

  – Second, the two stages may require different technical or managerial skills, may be located in different countries, or may be subject to different tax treatments. For these reasons one firm may sell a partially completed project to another. Our method lets us value a partially completed project.

  – Third, we will apply this approach to problems where completion of the project takes time.
Figure 10.1: Critical Prices and Option Values for Two-Stage Project.

- Extension to projects with three or more stages: Start at the end and work backwards, using solution for each stage in the boundary conditions for the previous stage. Value of option to invest in any stage $j$ of an $N$-stage project is of the form

$$F_j(P) = D_j \ P^{\beta_1},$$

and the coefficient $D_j$ and critical price $P^*_j$ are found by solving equations (11) and (12), with $P^*_j$ replacing $P^*_2$ and $I_j + I_{j+1} + \ldots + I_N$ replacing $I_2$. 

55
Cost Uncertainty and Learning

- Actual cost of completing project is a random variable, $\tilde{K}$; only the expected cost $K = \mathcal{E}(\tilde{K})$ is known.
- Project takes time to complete — maximum rate at which firm can (productively) invest is $k$.
- On completion, firm receives asset (e.g., factory or new drug) whose value, $V$, is known with certainty.
- Expected cost $K$ evolves according to
  \[ dK = -I \, dt + \nu (I \, K)^{1/2} \, dz \]  (56)
  - All risk associated with $dz$ is diversifiable.
  - Note that $K$ changes only if firm is investing, and variance of $dK/K$ increases linearly with $I/K$.
  - When firm is investing, expected change in $K$ over $\Delta t$ is $-I \, \Delta t$, but the realized change can be greater or less, and $K$ can even increase.
  - Variance of $\tilde{K}$ falls as $K$ falls, but total cost of project, $\int_0^T I \, dt$, only known when project is completed.
• Problem: Find investment policy that maximizes value of investment opportunity, $F(K) = F(K; V, k)$:

$$F(K) = \max \mathcal{E}_0 \left[ Ve^{-\mu \tilde{T}} - \int_0^{\tilde{T}} I(t)e^{-\mu t} \, dt \right]$$  \hspace{1cm} (57)$$

subject to eq. (56), $0 \leq I(t) \leq k$, and $K(\tilde{T}) = 0$. Here $\mu$ is risk-adjusted discount rate, and time of completion, $\tilde{T}$, is stochastic.

• Solution: You can confirm that $F(K)$ must satisfy

$$\frac{1}{2} \nu^2 I K F''(K) - I F'(K) - I = r F(K).$$  \hspace{1cm} (58)$$

- Eq. (58) is linear in $I$, so rate of investment that maximizes $F(K)$ is either zero or maximum rate $k$:

$$I = \begin{cases} 
  k & \text{for } \frac{1}{2} \nu^2 K F''(K) - F'(K) - 1 \geq 0 \\
  0 & \text{otherwise}
\end{cases}$$ \hspace{1cm} (59)$$

- There is a point $K^*$, such that $I(t) = k$ when $K \leq K^*$ and $I(t) = 0$ otherwise.
$K^*$ found with $F(K)$ by solving (58) subject to

$$ F(0) = V , \quad (60) $$

$$ \lim_{K \to \infty} F(K) = 0 , \quad (61) $$

$$ \frac{1}{2} \nu^2 K^* F''(K^*) - F'(K^*) - 1 = 0 , \quad (62) $$

and condition that $F(K)$ is continuous at $K^*$. 

- $K$ can change only when $I > 0$, so if $K > K^*$ and firm is not investing, $K$ will never change, and $F(K) = 0$.

- When $r = 0$, eq. (58) has analytical solution:

$$ F(K) = V - K + \nu^2 \left( \frac{V}{2} \right)^{-2/\nu^2} \left( \frac{K}{\nu^2 + 2} \right)^{(\nu^2+2)/\nu^2} , \quad (63) $$

and critical value $K^*$ is

$$ K^* = (1 + \frac{1}{2} \nu^2) V. $$
- Eq. (63) has simple interpretation. When \( r = 0 \), \( V - K \) would be value of investment opportunity were there no possibility of abandoning the project. The last term is the value of put option, i.e., option to abandon should cost turn out to be higher than expected.

- Note that for \( \nu > 0 \), \( K^* > V \), and \( K^* \) is increasing in \( \nu \). The more uncertainty there is, the greater the value of the investment opportunity, and the larger is the maximum expected cost for which beginning to invest is economical.

- Here, simple NPV might say don’t invest, but in fact you should invest.

• When \( r > 0 \), use numerical solution. For \( r = .05 \), \( V = 10 \), and \( k = 2 \), table shows \( K^* \) and \( F(K^*) \) for different values of \( \nu \).
<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$K^*$</th>
<th>$F(K^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.9257</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>8.9844</td>
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<td>9.7168</td>
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<td>10.156</td>
<td>0.5199</td>
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<td>10.693</td>
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<td>0.7</td>
<td>11.328</td>
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<td>0.8</td>
<td>12.051</td>
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</tr>
<tr>
<td>0.9</td>
<td>12.861</td>
<td>1.3606</td>
</tr>
<tr>
<td>1.0</td>
<td>13.770</td>
<td>1.6034</td>
</tr>
</tbody>
</table>

- Figs. 1 and 2 show $F(K)$ as a function of $K$ for three values of $\nu$, and for different values of $k$. 
Figure 1: Technical Uncertainty.

Figure 2: Changes in Maximum Rate of Investment.
How to estimate $\nu$ in practice? Make use of the fact that variance of cost to completion is

$$\mathcal{V}(\tilde{K}) = \left( \frac{\nu^2}{2 - \nu^2} \right) K^2.$$  \hspace{1cm} (64)

(Thus if one S.D. of a project’s cost is 25% (50%) of the expected cost, $\nu$ would be 0.343 (0.63).) Using (64) and an initial estimate of expected cost, $K(0)$, a value for $\nu$ can be based on an estimate of the S.D. of $\tilde{K}$. 


Generalization of Model

- In Dixit & Pindyck, Chap. 10, model also includes input cost uncertainty:
  \[ dK = -Idt + \nu(IK)^{1/2}dz + \gamma Kdw \]

  where \( dz \) and \( dw \) are increments of uncorrelated Wiener processes, and \( dw \) may be partly non-diversifiable.

- For nuclear power plant construction and abandonment decisions during the 1980s, \( \gamma Kdw \) term is crucial. See R. Pindyck, “Investments of Uncertain Cost,” in Journal of Financial Economics, 1993.

- Schwartz and Moon generalized model to also include
  - Asset value uncertainty. \( V \) evolves as
    \[ dV = \mu V dt + \sigma V dw \]
  - Catastrophic events. Possible arrival (Poisson process) of “catastrophic event” that drives \( V \) to zero.
  - They use numerical solution method, and apply model to drug development and FDA testing.
Schwartz-Moon Model

- Cost Uncertainty:

\[ dK = -Idt + \beta(IK)^{1/2}dz \]

where \( K = \mathcal{E}(\tilde{K}) \), and \( \tilde{K} \) is actual cost to completion.

\[ -\text{Var} (\tilde{K}) = \left( \frac{\beta^2}{2-\beta^2} \right) K^2 \]

\[ -I_m = k \] = maximum rate of investment.

- Asset Value Uncertainty:

\[ dV = \mu V dt + \sigma V d\omega \]

- “Catastrophic Event”: \( \lambda \) is probability (per year) that \( V \) jumps to 0.

- Investment Rule: Invest at maximum rate when \( V \geq V^*(K) \); \( I = 0 \) when \( V < V^*(K) \).
• Differential Equation (you derive as exercise):

When \( V > V^*(K) \), \( F(V, K) \) must satisfy
\[
\frac{1}{2}\sigma^2 V^2 F_{VV} + \frac{1}{2}\beta^2 I_m K F_{KK} + (\mu - \eta) V F_V \\
- I_m F_K - (r + \lambda) F - I_m = 0
\]

and when \( V < V^*(K) \), it must satisfy
\[
\frac{1}{2}\sigma^2 V^2 F_{VV} + (\mu - \eta) V F_V - (r + \lambda) F = 0
\]

Boundary conditions:
\[
F(V, 0) = V \quad (65) \\
F(0, K) = 0 \quad (66) \\
\lim_{K \to \infty} F(V, K) = 0 \quad (67) \\
\frac{1}{2}\beta^2 K F_{KK}(V^*, K) - F_K(V^*, K) - 1 = 0 \quad (68)
\]

and \( F(V, K) \) must be continuous at \( V^*(K) \).

• Fig. 3 shows \( V^*(K) \) for \( \mu = 0, \sigma = 0.35, \beta = 0.5, \lambda = 0.01, r = 0.05, \eta = 0.08, I_m = 20 \).
Figure 3: Critical Asset Values for Different Expected Completion Costs.
Extend model to 4 distinct phases: Phase I testing, Phase II testing, Phase III testing, FDA review.

- Each phase has different characteristics: amount of investment, maximum rate of investment, and probability of failure or catastrophic event during the phase.
- There may be differences in uncertainty related to cost to completion and to asset value obtained at completion.

Like a compound option: Completion of first phase gives option to start second phase, and so on.

- Boundary condition of each phase is the beginning value for the next phase.
- Problem is solved recursively starting from last phase, which has as its terminal boundary the asset value at completion.
- Penultimate phase has as its terminal boundary the value of the project before last investment phase is done, and so on.
Table 1 shows data.

<table>
<thead>
<tr>
<th></th>
<th>FDA</th>
<th>FDA</th>
<th>FDA</th>
<th>FDA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Phase I</td>
<td>Phase II</td>
<td>Phase III</td>
<td>review</td>
</tr>
<tr>
<td>Expected cost (million)</td>
<td>4</td>
<td>10</td>
<td>60</td>
<td>10</td>
</tr>
<tr>
<td>Maximum rate of investment/year (million)</td>
<td>2</td>
<td>5</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>Probability of failure ($\lambda$)</td>
<td>0.15</td>
<td>0.25</td>
<td>0.06</td>
<td>0.01</td>
</tr>
<tr>
<td>Asset return volatility ($\sigma$)</td>
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<td>0.35</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td>Drift of asset value process ($\mu$)</td>
<td>0.0</td>
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<td>0.0</td>
<td>0.0</td>
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<tr>
<td>Volatility of cost ($\beta$)</td>
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</tr>
<tr>
<td>Interest rate ($r$)</td>
<td>0.50</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>Risk premium ($\eta$)</td>
<td>0.08</td>
<td>0.08</td>
<td>0.08</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Table 2 shows $V^*$ at beginning of each stage.

<table>
<thead>
<tr>
<th></th>
<th>FDA</th>
<th>FDA</th>
<th>FDA</th>
<th>FDA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Phase I</td>
<td>Phase II</td>
<td>Phase III</td>
<td>review</td>
</tr>
<tr>
<td>Critical asset value ($V^*$)</td>
<td>250</td>
<td>215</td>
<td>180</td>
<td>15</td>
</tr>
<tr>
<td>Project value</td>
<td>300</td>
<td>5.7</td>
<td>13.9</td>
<td>45.2</td>
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<tr>
<td>Project volatility</td>
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<td>1.15</td>
<td>1.15</td>
<td>0.98</td>
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<tr>
<td>Project beta</td>
<td>300</td>
<td>3.06</td>
<td>2.82</td>
<td>2.13</td>
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<tr>
<td>Project value</td>
<td>500</td>
<td>20.2</td>
<td>43.3</td>
<td>110.9</td>
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<tr>
<td>Project volatility</td>
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<td>0.78</td>
<td>0.76</td>
<td>0.64</td>
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<tr>
<td>Project beta</td>
<td>500</td>
<td>2.00</td>
<td>1.80</td>
<td>1.50</td>
</tr>
</tbody>
</table>
Fig. 4 shows project value $F(V, K)$, as a function of $V$, at beginning of each stage.

\[ \text{Figure 4: Project Value at Each of Four Phases of New Drug Development.} \]