Sometimes uncertainty is discrete in nature.

- Competitor enters with better product, making yours worthless.
- New regulations make your factory worth less (or more).
- Sudden, unexpected success in the laboratory.
- Foreign operation is expropriated, or tax treatment changed.
- War, financial collapse, pestilence, etc.

As long as these discrete events are non-systematic (diversifiable), easy to handle.

Model as jump (Poisson) process, $dq$. Analogous to Wiener process:

$$dq = \begin{cases} 
0 & \text{with probability } 1 - \lambda dt \\
u & \text{with probability } \lambda dt.
\end{cases}$$

where $u$ is the size of the jump (and can be random).
Simple Example: Value of a Machine

• Suppose a machine produces constant flow of profit, $\pi$, as long as it operates.

• First, assume it lasts forever and never fails. No risk. Then asset return equation is:

$$rV \, dt = \pi \, dt$$

and value of machine is $V = \pi / r$.

• Now suppose at some point machine will break down and have to be discarded. So value of the machine follows the process:

$$dV - V \, dq$$

where $dq$ is a jump (Poisson) process. Now asset return equation is:

$$rV \, dt = \pi \, dt + \mathcal{E}(dV) = \pi \, dt - \lambda V \, dt$$

Thus,

$$V = \frac{\pi}{r + \lambda}.$$ 

So just increase discount rate by $\lambda$. 
Undeveloped Oil Reserve

Back to undeveloped oil reserve. Recall that value of developed reserve followed the process:

$$dV = (\mu - \delta)V dt + \sigma V dz$$

where $\delta$ is payout rate net of depletion:

$$\delta = \omega(\Pi - V)/V \approx 0.04$$

Now suppose a developed reserve is subject to full or partial expropriation. Then $V$ follows:

$$dV = (\mu - \delta)V dt + \sigma V dz - V dq$$

where $dq$ is a jump process with mean arrival rate $\lambda$, and $\mathcal{E}(dzdq) = 0$.

If “event” occurs, $q$ falls by a fixed percentage $\phi$ (with $0 \leq \phi \leq 1$). Thus $V$ fluctuates as a GBM, but over each $dt$ there is a small probability $\lambda dt$ that it will drop to $(1 - \phi)$ times its original value, and then continue fluctuating until another event occurs.
How to Estimate Arrival Rate $\lambda$?

- Begin with estimate of expected time $T$ for event to occur, e.g., 5 years.
- Now get equation for $\mathcal{E}(T)$. Probability that no event occurs over $(0, T)$ is $e^{-\lambda T}$. So probability that the first event occurs in the short interval $(T, T + dT)$ is $e^{-\lambda T} \lambda dT$. So expected time until $V$ jumps is:

$$\mathcal{E}[T] = \int_0^\infty \lambda T e^{-\lambda T} dT = \frac{1}{\lambda}$$

- So if expected $T$ is 5 years (60 months), use 0.2 (.01667) for $\lambda$. 
Optimal Investment Rule

• Want to find $F(V)$, value of undeveloped reserve, and optimal exercise point $V^*$. 

• As we saw earlier, the $dz$ component of $dV$ can be “replicated.” We assume $dq$ is non-systematic, i.e., can be diversified. So use risk-free rate. Then return equation is:

$$rF dt = \mathcal{E}(dF).$$

Expand $dF$:

$$r F dt = (\mu - \delta)VF'(V)dt + \frac{1}{2} \sigma^2 V^2 F''(V)dt - \lambda \{ F(V) - F[(1 - \phi) V] \} dt.$$

Can rewrite this as:

$$\frac{1}{2} \sigma^2 V^2 F''(V) + (r - \delta)VF'(V) - (r + \lambda)F(V) + \lambda F[(1 - \phi) V] = 0.$$

The same boundary conditions apply as before.
Optimal Investment Rule (Continued)

• Solution is again of the form $F(V) = AV^{\beta_1}$, but now $\beta_1$ is the positive solution to a slightly more complicated equation:

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + (r - \delta)\beta - (r + \lambda) + \lambda(1 - \phi) \beta = 0$$

Value of $\beta$ that satisfies this and also satisfies $F(0) = 0$ can be found numerically. Then $V^*$ and $A$ can be found.

• If $\phi = 1$ (so “event” is that $V$ falls to zero) above equation is a quadratic equation, and positive solution is:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)/\sigma^2 + \sqrt{[(r - \delta)/\sigma^2 - \frac{1}{2}]^2 + 2(r + \lambda)/\sigma^2}}$$

• Table 1 shows $\beta_1$, $V^*$, and $a$ for various values of $\lambda$, for case of $\phi = 1$. A positive value of $\lambda$ affects $F(V)$ in two ways.

  – First, it reduces the expected rate of capital gain on $V$ (from $\alpha$ to $\alpha - \lambda$), which reduces $F(V)$.

  – Second, it increases variance of changes in $V$, which increases $F(V)$. 
Table 1: Dependence of $\beta_1$, $V^*$, and $A$ on $\lambda$

(Note: $I = 1, \phi = 1, r = \delta = .04, \text{and } \sigma = .2$.)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\beta$</th>
<th>$V^*$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>2.00</td>
<td>.250</td>
</tr>
<tr>
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<tr>
<td>1.0</td>
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<td>1.15</td>
<td>.005</td>
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</tbody>
</table>

– As Table 1 shows, net effect is to reduce $F(V)$, and thus reduce the critical value $V^*$.

– This net effect is strong; small increases in $\lambda$ lead to big drop in $V^*$. 
• Note that we increased $\lambda$ while holding $\alpha = \mu - \delta$ fixed. Could argue that the market-determined expected rate of return on $V$ should remain constant, so that an increase in $\lambda$ is accompanied by a commensurate-rate increase in $\alpha$ (otherwise no investor would hold this project).

• Suppose $\phi = 1$. If $\alpha$ increases as much as $\lambda$ so $\alpha - \lambda$ is constant, we have to replace the terms $(r - \delta)$ in equation for $\beta$ with $(r + \lambda - \delta)$. Then an increase in $\lambda$ is like an increase in the risk-free rate $r$, and leads to an *increase* in $F(V)$ and $V^*$. 

• The simple jump process we used leads to a differential equation for $F(V)$ that is easy to solve. Could specify different process for $V$.
  
  – Firm holding a patent faces competitors, each trying to develop its own patent. Success of a competitor might cause $V$ to fall by a *random*, rather than fixed amount. Over time additional competitors may enter, so $V$ continues to fall.
  
  – Calculation of optimal investment rule is more difficult, and would require numerical solution method.