Asset Prices in a Heterogenous-Agent Economy with Portfolio Constraints

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Abstract

Our objective in this article is to study analytically the effect of borrowing constraints on asset returns in an economy where agents are heterogenous with respect to their risk aversion. We use asymptotic analysis to characterize the equilibrium in a general equilibrium exchange economy with an arbitrary number of agents who differ in their risk aversion and face limits on borrowing. We find that in the unconstrained economy the volatility of stock returns increases with the second moment of the cross-sectional distribution of risk aversion, while the risk-free rate and the equity premium are affected primarily by the first moment of the cross-sectional distribution of risk aversion. Limiting borrowing reduces the volatility of stock returns, lowers the risk-free interest rate and increases the equity risk premium. The method used for characterizing the equilibrium analytically applies also to other settings with a stochastic investment opportunity set and incomplete financial markets.

JEL classification: G12, G11, D52, C63.

Key words: Incomplete markets, borrowing constraints, portfolio choice, stochastic investment opportunities, asymptotic analysis.

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1 Introduction

Recent asset-pricing models have explored the effects on asset returns of heterogeneity in preferences to see if this can help explain the empirical properties of returns. For instance, Chan and Kogan (2002) find that in a model where agents differ in their risk aversion the time series properties of asset returns are closer to their empirical counterparts. Wang (1996) studies the effect of heterogeneity in risk aversion on bond yields, while Dumas (1989) explores the effect on the riskfree rate and the risk premium. In these models, heterogeneity in risk aversion gives rise to differences in portfolio holdings, and therefore, changes in the distribution of wealth over time. This is the key mechanism driving the dynamics of asset prices in such economies. However, these models assume that financial markets are complete and frictionless. Constraints on portfolio positions, such as limits on borrowing, are an important feature of the real economy that would change the cross-sectional distribution of individual asset holdings, and therefore, the evolution of the wealth distribution. Taking into account portfolio constraints is a challenging task since even in these frictionless models, most of the asset-pricing results are obtained using numerical analysis.\footnote{Wang can solve for only some of the quantities of the model in closed form and even this is possible only for particular combinations of the number of agents and the degree of risk aversion for each of these agents, while the analysis in Chan and Kogan is numerical. Similarly, the model in Dumas can be solved only numerically even when there are just two agents.} And, even in partial equilibrium models of the type studied in Liu (1998), Chacko and Viceira (1999), and Wachter (2002), it is not possible to characterize analytically the optimal weights in the presence of portfolio constraints. Our objective in this article is to study analytically the effect of borrowing constraints on asset returns in a general equilibrium economy where agents are heterogenous with respect to their risk aversion.\footnote{Cuoco (1997) considers heterogeneity in risk aversion but does not obtain an explicit solution. Examples of some other sources of heterogeneity considered in the literature are differences in beliefs (Detemple and Murthy, 1997), differences in labor income (Heaton and Lucas, 1996; Constantinides, Donaldson and Mehra, 2002), and differences in access to capital markets (Basak and Cuoco, 1998). To get explicit solutions, these papers either have to rely on numerical methods or need to assume that all investors have log utility. In contrast, we provide an explicit characterization in terms of exogenous variables for the consumption and portfolio policies and also for asset prices.}

We study a general equilibrium exchange economy with an arbitrary number of agents who differ in their risk aversion and face limits on borrowing. We use asymptotic analysis to characterize the equilibrium analytically, which allows us to establish an explicit link between the moments of asset returns and the moments of the cross-sectional distribution of risk aversion. We find that in the unconstrained economy the volatility of stock returns increases with the second moment of the...
cross-sectional distribution of risk aversion, while the risk-free rate and the equity premium are affected primarily by the first moment of the cross-sectional distribution of risk aversion. The effect of a constraint on borrowing is to increase the equity risk premium through a decrease in the risk-free interest rate; however, this is associated with a decrease in the volatility of stock returns. While Basak and Cuoco (1998) and Detemple and Murthy (1997) obtain results for the first moment of returns, they can characterize the second moment of returns only in a setting where all agents have log utility, and hence, volatility does not change with the imposition of a constraint on portfolio positions.

In addition to the analysis of the effects of borrowing constraints on asset returns, a second contribution of our paper is to provide a method for analyzing analytically the equilibrium policies and prices in an economy with a stochastic investment opportunity set and incomplete financial markets. Identifying the equilibrium in multiagent economies with incomplete financial markets is a difficult problem and to date the literature does not have an explicit general characterization in terms of exogenous variables.\(^3\) Cuoco and He (1994a,b) show that with incomplete markets one can still construct a representative agent, but in this case the weights assigned to individual agents in this aggregation evolve stochastically. However, the characterization of equilibrium in Cuoco and He is in terms of endogenous variables, while our method allows for an explicit characterization in terms of exogenous variables. The method developed in this paper can also be applied to more general settings. For instance, we have assumed that agents have time-additive power utility but the method applies to the Duffie and Epstein (1992) recursive preferences—as demonstrated in Trojani and Vanini (2002), and to preferences that exhibit habit-persistence—as shown in Chan and Kogan (2002). Moreover, while the model studied in the paper is of a general equilibrium exchange economy, the method applies also to a production economy (such as the one considered in Dumas, 1989) and these results are available from the authors.

Besides the general equilibrium models described above, our method also applies to partial equilibrium models of intertemporal portfolio choice with a stochastic investment opportunity set (Merton, 1971). Recent papers studying this problem include Kim and Omberg (1996), Liu (1998), Chacko and Viceira (1999), Xia (2001), Wachter (2002), and Detemple, Garcia and Rindisbacher\(^3\) For example, Telmer (1993), Heaton and Lucas (1996), and Marcet and Singleton (1999) use numerical methods to solve for the equilibrium in such economies.
In contrast to these papers, our approach does not need to assume that markets are complete or that agents derive utility only from terminal wealth.

Our solution method relies on asymptotic analysis, which allows us to obtain in closed-form the approximate (asymptotic) expressions for portfolio and consumption policies. The basic idea of asymptotic methods is to formulate a general problem, find a particular case that has a known solution, and use this as a starting point for computing the solution to nearby problems. In the context of portfolio problems, the solution for the investor with log utility (with unit risk aversion) provides a convenient starting point for the expansion. We need to emphasize, though, that while our method allows for exact comparative statics results around the case of log utility, it provides only approximations to the portfolio rules and asset prices. Thus, it should be viewed as being complementary to numerical methods rather than a substitute, and we compare our analytical results with the numerical solution of an economy with two agents.

The rest of the paper is arranged as follows. In Section 2, we describe an exchange economy with heterogenous agents and leverage constraints. In Section 3, we characterize the equilibrium in this economy using analytic and numerical methods. We conclude in Section 4. Our main results are highlighted in propositions and the proofs for all the propositions are collected in the appendix.

2 A model of an exchange economy with heterogenous agents

In this section, we study a general-equilibrium exchange (endowment) economy with multiple agents who differ in their level of risk aversion. Wang (1996) analyzes this economy for the case where there are two agents who do not face any portfolio constraints.\footnote{Wang (1996) also discusses how the model could be solved when there are up to 4 agents, each having a particular degree of risk aversion; with more than 4 agents a closed-form solution is not available for general wealth distributions.} We extend the analysis of Wang in several directions. First, we analyze the equilibrium for the case where there is an arbitrary number of agents. In contrast to Wang, we also obtain closed-form (asymptotic) expressions for the mean and volatility of the stock return process. This analysis allows us to relate the volatility of stock returns to the heterogeneity of investors in their degree of risk aversion and to the cross-sectional

\footnote{While the asymptotic solution is designed to provide a local approximation (for risk aversion close to unity), general theoretical results on the magnitude of the approximation error are not available—see Judd (1996, 1998, Ch. 13–15) for a discussion of these issues. However, there do exist a number of methods to evaluate the quality of the approximate solution numerically (for instance, see Den Haan and Marcet, 1994, and Judd, 1996 and 1998) that can also be applied to the problem considered here.}
dispersion in stock holdings. In the second part of this section, we study the effect of introducing a leverage constraint that restricts how much investors can borrow to lever their investment in the stock.

### 2.1 The endowment process

The infinite-horizon exchange economy has an aggregate endowment, $e_t$, that evolves according to

$$de_t = \mu_e e_t dt + \sigma_e e_t dZ_t,$$

where $\mu_e$ and $\sigma_e$ are constant parameters. We assume that the growth rate of the endowment is positive, $\mu_e - \frac{\sigma_e^2}{2} > 0$.

### 2.2 Preferences

The utility function of an agent is time-separable and is given by

$$E_0 \left[ \int_0^\infty e^{-\rho t} \frac{1}{\gamma} (C_t^\gamma - 1) \ dt \right],$$

where $\rho$ is the constant subjective time discount rate, and $C_t$ is the flow of consumption. The agent’s relative risk aversion is given by $1 - \gamma$, and for agents with unit risk aversion ($\gamma = 0$), utility is given by the logarithmic function:

$$E_0 \left[ \int_0^\infty e^{-\rho t} \ln C_t \ dt \right].$$

In order to allow for multiple agents who differ in their risk aversion, we define $\gamma \equiv \epsilon a$, where $a$ is used to index agent types so that differences in $a$ lead to differences in risk aversion, while the parameter $\epsilon$ allows us to set the magnitude of these differences. Without loss of generality, assume that there is a single agent of each type $a$.

### 2.3 Financial assets

We assume that there are two assets available for trading in the economy. The first asset is a short-term risk-free bond, available in zero net supply, which pays the interest rate $r_t$ that will be
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The second asset is a stock that is a claim on the aggregate endowment. The price of the stock, $P_t$, evolves according to

$$\frac{dP_t}{P_t} = \mu_P dt + \sigma_P dZ_t,$$

where $\mu_P$ is the instantaneous expected return and $\sigma_P$ is the volatility. The cumulative stock return process is

$$\frac{dP_t + e_t dt}{P_t} = \mu_R dt + \sigma_R dZ_t,$$

with $\mu_R$ and $\sigma_R$ to be determined in equilibrium.

Let the investment opportunity set be described by the vector of state variables, $X_t$. In equilibrium the state vector changes over time according to

$$dX_t = \mu_X (X_t) dt + \sigma'_X (X_t) \cdot dZ_{Xt},$$

with the covariance between the stock returns process and the state vector process denoted by $\sigma_{PX}$. With the above specification, the riskless rate and the expected rate of return and volatility of the risky asset may depend on the state vector:

$$r_t = r(X_t), \quad \mu_P = \mu_P(X_t), \quad \sigma_P = \sigma_P(X_t),$$

implying that the instantaneous market price of risk is also stochastic:

$$\phi_t = \phi(X_t) \equiv \frac{\mu_P - r_t}{\sigma_P^2}.$$

### 2.4 Consumption and portfolio policies in the absence of portfolio constraints

In the above economy, denoting by $\pi_t$ the proportion of the agent’s wealth invested in the risky asset, the wealth of the agent evolves according to

$$dW_t = [(r_t + \pi_t(\mu_P - r_t)) W_t - C_t] dt + \pi_t \sigma_P W_t dZ_{Pt}.$$

The value function $J(W_t, X_t)$ of the optimal control problem is defined by

$$J(W_t, X_t) = \sup_{\{C_s, \pi_s\}} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \frac{1}{\gamma} (C_s^\gamma - 1) \, ds \right],$$
subject to equations (1), (3), and (4). Defining the consumption-wealth ratio \( c \equiv C/W \), the function \( J(W, X) \), satisfies the Hamilton-Jacobi-Bellman equation

\[
0 = \max_{c, \pi} \left\{ \frac{1}{\gamma} ((Wc)^\gamma - 1) - \rho J + (r + \pi(\mu_P - r_t) - c) JW W + \frac{1}{2} \pi^2 W^2 JW W \sigma_P^2 \right. \\
+ \mu'_X \cdot J_X + \frac{1}{2} \sigma'_X \cdot J_{XX} \cdot \sigma_X + \pi W \sigma_P^2 \cdot J_W X
\]

Given the homogeneity of the utility function, the solution to this equation has the following functional form:

\[
J(W, X) = \left( e^{g(X)W} \right)^{\gamma} - 1.
\]

The exact solution for the optimal consumption policy and portfolio weight can be obtained from the first-order conditions implied by the Hamilton-Jacobi-Bellman equation:

\[
c(X) = \left( \frac{1}{\rho} \right)^{1/(\gamma-1)} e^{\gamma g(X)},
\]

\[
\pi(X) = \frac{J_W}{W JW W} \phi(X) - \frac{J_W}{W JW W} \frac{J_W X \sigma'_P(X)}{J_W \sigma_P^2(X)}
\]

\[
= \frac{1}{1 - \gamma} \phi(X) + \frac{\gamma \sigma'_P(X)}{1 - \gamma} \frac{\partial g(X)}{\partial X},
\]

where the second line is obtained by using (5).

Observe from equations (6) and (7) that \( c_t \) and \( \pi_t \) depend on the unknown function \( g(X) \). In general, the unknown function \( g(X) \) cannot be computed in closed form. We describe in Section 3 how one can obtain an asymptotic approximation to \( g(X) \) with respect to the risk aversion parameter, \( \epsilon \), and then explain how this can then be used to characterize the equilibrium.

### 2.5 The constraint on borrowing

Up to this point, the agent’s portfolio choice was unconstrained. We now extend the model to allow for constraints on the portfolio weights. We consider a leverage constraint that restricts the portfolio weight on the risky asset to lie below an upper bound:

\[
\pi(X) \leq \pi,
\]
This specification of portfolio constraints is a specialization of the formulation in Cvitanic and Karatzas (1992) to the case of one risky asset. By restricting our attention to the constraint on portfolio proportions, we are ruling out more general types of constraints, e.g., the constraints on the absolute amount invested in each asset (see Grossman and Vila (1992), Cuoco (1997)).

### 2.6 Equilibrium in the economy

The equilibrium in this economy is defined by the stock price process, \( P_t \), the interest rate process \( r_t \), and the portfolio and consumption policies, such that (i) given the price processes for financial assets, the consumption and portfolio choices are optimal for the agents, (ii) the goods market and the markets for the stock and the bond clear.

Defining the wealth of individual agents by \( W(a) \) and the distribution of wealth by \( \omega(a) = W(a)/\sum_a W(a) \), the conditions for market clearing in the stock and commodity markets are:

\[
\sum_a \pi_t(a) \omega_t(a) = 1, \tag{9}
\]

\[
\sum_a C_t(a) = \sum_a c_t(a) W_t(a) = e_t, \tag{10}
\]

where, if one wishes to consider a continuum of agents, the summation signs should be replaced by integrals.

### 3 Characterizing the equilibrium in the economy

In this section, we wish to study the equilibrium in the economy described above. We start by describing the asymptotic method that we will use to characterize analytically the optimal consumption and portfolio policy and the asset prices. Then, we apply this method to the economy without portfolio constraints in order to understand the effect of heterogeneity. Following this, we study the economy with a borrowing constraint. This model can no longer be solved using the representative-agent approach used in Wang (1996), and thus, we rely on the asymptotic method to characterize the solution. We conclude by comparing the asymptotic solution to the exact solution obtained numerically.
3.1 An asymptotic approach to consumption and portfolio choice in equilibrium

In Section 3.1.1, we describe the asymptotic consumption and portfolio rules for an arbitrary stochastic vector process for the state variables that drives changes in the investment opportunity set. In Section 3.1.2, we derive the consumption and portfolio rules with borrowing constraints, and in Section 3.1.3 we obtain the policies in a general equilibrium setting. The application of these results to the particular economy we wish to study is done in Section 3.2 and Section 3.3.

3.1.1 Asymptotic consumption and portfolio policies in the absence of constraints

Recall that an agent's risk aversion is defined as $\gamma \equiv \epsilon a$, where $a$ is used to index agent types, while the parameter $\epsilon$ determines the magnitude of these differences. In order to identify the consumption and portfolio rules given in equations (6) and (7), we look for $g(X)$ as a power series in $\epsilon$:

$$g(X) = g_0(X) + \epsilon g_1(X) + O(\epsilon^2),$$

(11)

where $g_0(X)$ is obtained from the value function of an agent with logarithmic utility ($\epsilon = 0$):

$$J(W, X) = \frac{1}{\rho} \left( \ln W(t) + g_0(X) \right).$$

Note that the first-order asymptotic expansions are sufficient to obtain local comparative statics results for the dependence of the optimal policies on the risk aversion parameter. The asymptotic expansions will also approximate the optimal consumption and portfolio policies when the risk aversion parameter $\gamma$ is sufficiently close to zero (that is, when $\epsilon$ is close to zero).

We now derive the asymptotic expansions for the consumption-portfolio problem (by substituting (11) into (7)) and explain how one can obtain the function $g_0(X)$.

Proposition 1 The first-order asymptotic expansions for the optimal consumption and portfolio choice are

$$c(X) = \rho - \epsilon a \rho \left( g_0(X) - \ln(\rho) \right) + O(\epsilon^2),$$

(12)

$$\pi(X) = \frac{1}{1 - \epsilon a} \phi(X) + \frac{\epsilon a}{1 - \epsilon a} \frac{\sigma^2_{p\mu}(X)}{\sigma^2_p(X)} \frac{\partial g_0(X)}{\partial X} + O(\epsilon^2),$$

(13)
with an asymptotically equivalent expression for the portfolio choice being

\[
\pi(X) = \phi(X) + \epsilon a \left( \phi(X) + \frac{\sigma_P'(X) \partial g_0(X)}{\sigma_P(X)} \right) + O(\epsilon^2),
\]

(14)

where the function \(g_0(X)\) is

\[
g_0(X) = \ln \rho - 1 + E_0 \left[ \int_0^\infty e^{-\rho t} \left( r(X_t) + \frac{\phi(X_t)^2 \sigma_P(X_t)^2}{2} \right) dt \bigg| X_0 = X \right].
\]

(15)

The two expressions for the portfolio weight, (13) and (14), are equally easy to manipulate. The role of the risk aversion coefficient is more apparent in (14), while (13) retains the exact form of the myopic portfolio demand, expanding only the hedging demand.

Comparing the asymptotic weight in (13) to the exact one in (7), we see that the only difference is that under the standard approach one needs to identify the unknown function \(g(X)\), while in our approach one needs to identify only \(g_0(X)\), a term in the value function for the log investor. It is much easier to solve for \(g_0(X)\) from equation (15), and as long as the function \(g_0(X)\) is known in closed form, one can obtain explicit first-order asymptotic expressions for the optimal consumption and portfolio policies. For example, the class of affine processes will yield closed-form solutions.

Analyzing the consumption-portfolio rules given in Proposition 1, we see that the zero-order components of these expansions correspond to the well-known solution for the case where the agent has a logarithmic utility function (\(\epsilon = 0\)): the optimal consumption-wealth ratio, \(c = C/W\), is given by \(\rho\), and the optimal portfolio policy is myopic and independent of changes in the investment opportunity set. The first-order terms capture the effect of risk aversion when the coefficient of relative risk aversion deviates from one (\(\epsilon\) deviates from zero). In particular, one can interpret the expression for the optimal portfolio in (13) as

\[
\pi(X) = \mathbf{myopic \ demand} + \mathbf{hedging \ demand} + O(\epsilon^2),
\]

where the first bracketed term represents the portfolio weights under constant investment opportunity set, the myopic demand, and the second term characterizes the demand arising from the desire
to hedge against intertemporal changes in the investment opportunity set. The important thing to note in the above expression is that it relies on \( g_0(X) \), which can be determined explicitly, rather than on \( g(X) \), which cannot be identified generally.

### 3.1.2 Asymptotic consumption and portfolio policies with portfolio constraints

In the presence of the portfolio constraint in (8), the value function of the agent’s constrained optimization problem now satisfies

\[
0 = \max_{\{c, \pi \leq \pi\}} \left\{ \frac{1}{\epsilon} \left( (Wc)^{\epsilon_a} - 1 \right) - \rho J + (r + \pi \phi \sigma_P^2 - c) J_W W + \frac{1}{2} \pi^2 W^2 J_W W \sigma_P^2 + \mu'_X \cdot J_X + \frac{1}{2} \sigma'_X \cdot J_X X \cdot \sigma_X + \pi W \sigma_P' \cdot J_W X \right\}.
\]

We then have the following result about the consumption and portfolio policies in the presence of this constraint.

**Proposition 2** *In the presence of constraints, the optimal portfolio choice is given by

\[
\pi(X) = \begin{cases} 
\tilde{\pi}(X), & \tilde{\pi}(X) \leq \pi, \\
\pi, & \tilde{\pi}(X) > \pi,
\end{cases}
\]

where

\[
\tilde{\pi} \equiv \phi(X) + \epsilon a \left( \phi(X) + \frac{1}{\sigma_P^2(X)} \sigma'_{P(X)}(X) \cdot \frac{\partial g_0^c(X)}{\partial X} \right) + O(\epsilon^2).
\]

and the optimal consumption policy is given by

\[
e(X) = \rho - \epsilon a \rho (g_0^c(X) - \ln(\rho)) + O(\epsilon^2),
\]

where

\[
g_0^c(X) = \ln \rho - 1 + \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( r(X_t) + \pi_0(X_t) \phi(X_t) \sigma_P(X_t)^2 - \frac{1}{2} \pi_0(X_t)^2 \sigma_P^2(X_t) \right) dt \right] X_0 = X,
\]

and

\[
\pi_0(X_t) = \begin{cases} 
\phi(X_t), & \phi(X_t) \leq \pi, \\
\pi, & \phi(X_t) > \pi.
\end{cases}
\]
The function $g_0^c(X)$ in (16), where the superscript “c” indicates the presence of constraints, is the counterpart of the function $g_0(X)$ in (13). As in the unconstrained case, an explicit asymptotic expression for the optimal consumption and portfolio policies is available as long as the solution of the analogous problem for the agent with the logarithmic utility function is known in closed form.

### 3.1.3 Asymptotic consumption and portfolio policies in general equilibrium

The results in Propositions 1 and 2 can be used as building blocks in the analysis of a broad range of models. In particular, they allow one to obtain asymptotic expressions for the prices of assets in equilibrium economies that otherwise can only be studied numerically.

In the heterogeneous-agent economy of Section 2, the investment opportunity set depends on the cross-sectional distribution of wealth among the agents. Thus, the state vector is given by

$$X_t = \{W_t(a)\},$$

where $\{W_t(a)\}$ is the set of individual wealth values for the agents in the economy, and $\gamma \equiv \epsilon a$ so that $a$ controls the distribution of risk aversion in the economy.

In general equilibrium, the moments of asset returns depend on the cross-sectional distribution of risk aversion. Therefore, we can approximate them by an asymptotic power series in $\epsilon$. Specifically, a moment of returns $m_t$ can be expressed as

$$m_t = m(X_t, \epsilon) = m_0 + \epsilon m_1(X_t) + O(\epsilon^2). \quad (20)$$

The leading term in the expansion coincides with the corresponding moment in an economy with $\epsilon = 0$, that is, in a homogeneous-agent economy with log-utility maximizing agents.

Next, consider the term $g_0(X, \epsilon)$ that characterizes the value function of the log-utility maximizer in our heterogeneous-agent economy. Since the moments of returns depend on $\epsilon$, so does the function $g_0(X, \epsilon)$, according to Proposition 1. Thus,

$$g_0(X, \epsilon) = g_{0,0} + \epsilon g_{0,1}(X) + O(\epsilon^2), \quad (21)$$

where the leading term, $g_{0,0}$, corresponds to the value function in the homogeneous-agent economy with log-utility maximizing agents and hence does not depend on the wealth distribution. From
(12) and (14) for the unconstrained case, or (17) and (16) for the case with constraints, we see that as long as one can identify the leading term $g_{0,0}$, one will be able to characterize the two leading terms in the expression for portfolio and consumption policies in a heterogeneous-agent economy.

3.2 Equilibrium in the economy without constraints

We can now use the results developed above to obtain asymptotic expansions for the individual consumption and portfolio policies in a heterogeneous-agent general equilibrium economy, where the state vector is: $X_t = \{W_t(a)\}$.

**Proposition 3** The optimal consumption and portfolio policies in a heterogeneous-agent economy are given by:

$$
\begin{align*}
  c(X) &= \rho - \epsilon a \rho \left(g_{0,0} - \ln \rho\right) + O\left(\epsilon^2\right), \\
  \pi(X) &= \phi(X) + \epsilon a \phi(X) + O\left(\epsilon^2\right).
\end{align*}
$$

where $g_{0,0}$ is given in (21) and $\phi(X)$ will be determined in equilibrium.

We start by identifying $g_{0,0}$, which characterizes the value function of the agents in an economy where $\epsilon = 0$ (that is, all agents have log utility). In such an economy, the value function of a representative agent equals the lifetime discounted utility from consuming the entire endowment stream

$$
\begin{align*}
  \frac{1}{\rho} \left( \ln W_0 + g_{0,0} \right) &= E_0 \left[ \int_0^\infty e^{-\rho t} \ln e_t dt \right]. \\
  &= \frac{1}{\rho} \left( \ln e_0 + A \right),
\end{align*}
$$

where

$$
A \equiv \frac{1}{\rho} \left( \mu_e - \frac{\sigma_e^2}{2} \right).
$$

Also, it must be the case that the aggregate wealth in the economy is equal to the price of the stock, which in a log-utility economy is:

$$
W_0 = \frac{1}{\rho} e_0.
$$
Thus, solving equation (22) for $g_{0,0}$, we have

$$g_{0,0} = \ln \rho + A.$$  

Substituting the above value for $g_{0,0}$ into Proposition 3 then leads to the consumption and portfolio policies:

$$c_t(a) = \rho - \epsilon a \rho A + O(\epsilon^2) \quad (23)$$

$$\pi_t(a) = (1 + \epsilon a) \phi_t + O(\epsilon^2), \quad (24)$$

where the market price of risk, $\phi_t = (\mu_R t - r_t) / \sigma_R^2$, needs to be determined in equilibrium.

Using the market-clearing conditions along with the expressions for the optimal consumption and portfolio policies for individual investors, and defining the first and second moments of the cross-sectional dispersion in the parameter $a$ that drives the differences in risk aversion by

$$E_a[a] \equiv \sum_a a \omega_t(a), \quad \text{var}_a[a] \equiv E_a[a^2] - (E_a[a])^2,$$

we have the following characterization of the equilibrium in the unconstrained economy.

**Proposition 4** For the exchange economy described above, in equilibrium:

(i) The price-dividend ratio is given by

$$\frac{P_t}{e_t} = \frac{1}{\rho} + \epsilon \frac{1}{\rho} A E_a[a] + O(\epsilon^2), \quad (25)$$

while the moments of the cumulative return process are

$$\mu_{Rt} = (\mu_e + \rho) - \epsilon \rho A E_a[a] + O(\epsilon^2), \quad (26)$$

$$\sigma_{Rt} = \sigma_e + \epsilon^2 A \sigma_e \text{var}_a[a] + O(\epsilon^3). \quad (27)$$

(ii) The interest rate is given by

$$r_t = (\mu_e - \sigma_e^2 + \rho) + \epsilon (\sigma_e^2 - \rho A) E_a[a] + O(\epsilon^2). \quad (28)$$

(iii) The optimal portfolio policy is

$$\pi_t(a) = 1 + \epsilon (a - E_a[a]) + O(\epsilon^2). \quad (29)$$
(iv) The cross-sectional wealth distribution evolves according to

$$\frac{d\omega_t(a)}{\omega_t(a)} = \epsilon \rho A (a - E_a[a]) \, dt + \epsilon \sigma_e (a - E_a[a]) \, dZ_t + O(\epsilon^2).$$

Observe that, to first order, the expected return on the stock in equation (26) and the riskless rate in equation (28) depend only on the first moment of the cross-sectional distribution of the parameter determining risk aversion in the economy; these expressions are asymptotically the same as in a homogenous-agent economy with risk aversion equal to $E_a[a]$. On the other hand, the volatility of stock returns in equation (27) is related to the second moment.

**Proposition 5** Asymptotically, the volatility of stock returns is increasing in the cross-sectional heterogeneity of risk aversion.

To understand the intuition behind this result, consider the price-dividend ratio in equation (25). The price-dividend ratio is decreasing in average risk aversion. This is because the expected stock return is increasing in average risk aversion, as shown in (26). Moreover, the average risk aversion in the economy fluctuates over time in response to the aggregate endowment shocks. According to (24), agents with relatively high risk aversion are less exposed to the stock market risk. Therefore, the fraction of total wealth controlled by agents with higher-than-average risk aversion declines as the stock market rises, as shown in (30). As a result, the average risk aversion in the economy is negatively affected by the aggregate endowment shocks, implying a positive effect on the price-dividend ratio.\(^6\) The positive impact of the endowment shocks on the price-dividend ratio increases the volatility of stock returns. Because the outlined effect is due to the cross-sectional differences in investors’ risk aversion, it is natural that its magnitude is related to the cross-sectional dispersion of individual types, as captured by (27).

Given that individual risk aversion coefficients are not directly observable, it is useful to re-state Proposition 5 in terms of individual portfolio choices. Because $\text{var}_a[\pi_t(a)] = \epsilon^2 \text{var}_a[a] + O(\epsilon^3)$, there

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\(^6\)Chan and Kogan (2002) discuss the countercyclical nature of expected stock returns due to investor heterogeneity in a setting were individuals have catching-up-with-the-Joneses preferences. Our setting uses a more common specification of individual preferences, but the same intuition for time-variation in expected returns applies in both cases. While Chan and Kogan rely on numerical analysis and focus on the dynamics of conditional moments of stock returns, we derive an explicit asymptotic relation between the level of return volatility and the degree of cross-sectional heterogeneity.
exists a positive linear asymptotic relation between the volatility of stock returns and the cross-sectional dispersion of individual portfolio holdings.

3.3 Equilibrium in the economy with constraints

In this section, we show that tightening the limit on borrowing has a negative effect on the volatility of stock returns while it lowers the risk-free interest rate and leaves the expected stock returns asymptotically unchanged.

In our asymptotic analysis, we will specify the constraint in (8) to be of the form

$$\pi(X) \leq \pi = 1 + \epsilon L,$$

That is, the constraint on leverage, $L$, is set so that it is proportional to the parameter $\epsilon$. This is because the equilibrium amount of borrowing in the unconstrained economy is proportional to $\epsilon$; therefore, in order for the leverage constraint to have an impact it must be sufficiently tight—formally, it must be proportional to $\epsilon$ as well.

From Proposition 2, the portfolio weight in the presence of constraints is given by:

$$\pi_t(a) = \min \left[ (1 + \epsilon a) \phi_t, 1 + \epsilon L \right] + O(\epsilon^2)$$

$$= 1 + \epsilon \min [a + \phi_{1t}, L] + O(\epsilon^2),$$

where,

$$\phi_t = \phi_{0t} + \epsilon \phi_{1t} + O(\epsilon^2) = 1 + \epsilon \phi_{1t} + O(\epsilon^2),$$

and the leading term in the expansion, $\phi_{0t}$, is the market price of risk in the economy populated with log-utility agents, which is equal to unity. This implies that the market-clearing condition for stocks is:

$$1 = \sum_a \pi_t(a) \omega_t(a)$$

$$= 1 + \epsilon \sum_a \min [a + \phi_{1t}, L] \omega_t(a) + O(\epsilon^2).$$

To analyze the equilibrium in the constrained economy, we first consider the case where there are only two classes of agents, $a'$ and $a''$, with $a' < a''$. Proceeding as in the unconstrained economy
case, we first identify $g_{0,0}$, then the optimal consumption and portfolio policies, and finally use the market clearing conditions to obtain the price processes.

**Proposition 6** In equilibrium, the leverage constraint is binding when $\omega_t(a')(a'' - a') - L \geq 0$. In this region:

(i) The price-dividend price is given by

$$\frac{P_t}{e_t} = \frac{1}{\rho} + \epsilon \frac{1}{\rho} A E_a[a] + O(\epsilon^2),$$

while the moments of the cumulative return process are

$$\mu_{Rt} = (\mu_e + \rho) - \epsilon \rho A E_a[a] + O(\epsilon^2),$$

$$\sigma_{Rt} = \sigma_e + \epsilon^2 A \sigma_e L [a'' - a'] \omega_t(a'') + O(\epsilon^3).$$

(ii) The interest rate is

$$r_t = (\mu_e - \sigma_e^2 + \rho) + \epsilon \left( \sigma_e^2 \left( L \frac{\omega_t(a'')}{\omega_t(a')} + a' \right) - \rho A E_a[a] \right) + O(\epsilon^2).$$

(iii) The portfolio policy is

$$\pi_t(a) = 1 + \epsilon \min \left[ a - L \frac{\omega_t(a'')}{\omega_t(a')} - a', L \right] + O(\epsilon^2).$$

(iv) The cross-sectional wealth distribution for the two types of agents evolves according to:

$$\frac{d\omega_t(a')}{\omega_t(a')} = \epsilon \rho A (a' - E_a[a]) \, dt - \epsilon \sigma_e L \frac{\omega_t(a'')}{\omega_t(a')} \, dZ_t + O(\epsilon^2),$$

$$\frac{d\omega_t(a'')}{\omega_t(a'')} = \epsilon \rho A (a'' - E_a[a]) \, dt + \epsilon \sigma_e L \, dZ_t + O(\epsilon^2).$$

When the leverage constraint is not binding, the solution is asymptotically the same as in the unconstrained case.

We now relate the volatility of stock returns to the leverage constraint and compare it to the volatility in the unconstrained economy.
Proposition 7  Asymptotically, the volatility of stock returns in the constrained economy with two agents is lower than in the unconstrained economy:

\[ \sigma^c_{R_t} \leq \sigma^u_{R_t} + O(\epsilon^3), \]

where \( \sigma^c_{R_t} \) and \( \sigma^u_{R_t} \) denote the volatility of stock returns in the constrained and the unconstrained economy, respectively. Moreover, the volatility of stock returns is reduced by tightening the borrowing constraint.

As we argued in the case of the unconstrained economy, the volatility of stock returns is positively related to the variability of the average risk aversion in the economy. The leverage constraint reduces the cross-sectional differences in individual portfolio holdings, and hence, the variability of the cross-sectional wealth distribution and the average risk aversion. As a result, the constraint on borrowing lowers the volatility of stock returns.

Next, we analyze the effect of the borrowing constraint on the interest rate and the equity premium. From equation (36), imposing the leverage constraint lowers the risk free interest rate. Formally, the difference between interest rates in the unconstrained and the constrained economies, given by

\[ \epsilon \sigma^2 e^t \omega_t(a''') \left( \frac{\omega_t(a')(a'' - a') - L}{\omega_t(a')} \right) + O(\epsilon^2), \]

is asymptotically positive, because \( \omega_t(a')(a'' - a') - L \geq 0 \) whenever the leverage constraint is binding. The expected stock return is asymptotically unaffected, according to (34). Thus, tightening the leverage constraint increases the equilibrium equity premium, \( \mu_{R_t} - r_t \).

We now explain the intuition for this result. Start by considering the situation where trading in the stock is not allowed in equilibrium. Then, one class of agents would borrow to increase their current consumption, thereby reducing the growth rate of their consumption. Hence, imposing the borrowing constraint would reduce current consumption and increase the consumption growth rate of the constrained agent while reducing the growth rate for the unconstrained agent. This

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7Heaton and Lucas (1996) observe similar behavior of asset returns in their incomplete-market model in response to an increased difference in borrowing and lending rates. Their analysis relies on numerical simulations and the intuition behind their results is different. In their model, individuals have the same risk aversion but face idiosyncratic endowment shocks. As a result, an increase in trading costs raises individual consumption variability, and hence, lowers the risk-free rate of return due to the demand for precautionary savings.
would lower the equilibrium interest rate, which is linked to the consumption growth rate of the unconstrained agent.

Now, consider the situation where agents can trade also the risky asset. This complicates matters in general. Intuitively, holding the asset price processes fixed, adding the leverage constraint reduces the aggregate demand for the risk-free asset, which suggests that tightening the leverage constraint would result in a lower interest rate in equilibrium. However, this argument ignores the potential impact of the constraint on stock returns. We find that the leverage constraint has only a higher-order effect on the moments of stock returns in our model, as can be seen from comparing Propositions 4 and 6. To understand the reason for this, note that the stock price is determined by market clearing for the consumption good, and the consumption policy (the consumption to wealth ratio) itself is affected by the leverage constraint only through its effect on the investment opportunity set. Because the consumption policy of investors with logarithmic preferences is independent of the investment opportunity set, for agents with utility functions close to logarithmic, the impact of changes in the investment opportunity set on the consumption policy is of order $\epsilon$ (see Proposition 2). Moreover, the time-varying component of the investment opportunity set is itself of order $\epsilon$ in equilibrium, since the economy is perturbed around the logarithmic representative agent case where the investment opportunities are constant over time. Therefore, these two effects imply that the impact of the leverage constraint on consumption policies is of order $\epsilon^2$ (see Proposition 3). Thus, the constraint has only a second-order effect on the equilibrium price-dividend ratio and the moments of stock returns.

The above argument explains the impact of the leverage constraint on the risk-free rate and the equity premium in our model. We summarize these observations in the following proposition.

**Proposition 8** Asymptotically, in the constrained economy with two agents, the interest rate is lowered and the equity premium is increased by tightening the borrowing constraint.

So far, we carried out our analysis for the special case of only two types of agents. We now show that the main results from the analysis of the two-agent economy generalize to an economy with a continuum of agents. Suppose that the continuum of agents have risk aversion between $-\bar{\alpha}$ and $+\bar{\alpha}$ and an arbitrary cross-sectional distribution of wealth. Then, we have the following result.
Proposition 9 Asymptotically, in the constrained economy the interest rate is lowered, the equity premium is increased, and the volatility of stock returns is reduced by tightening the borrowing constraint.

3.4 Comparison of the asymptotic solution with the exact numerical solution

In order to demonstrate that the analytical results described above capture the salient qualitative features of the exact solution, we solve numerically the unconstrained model for the case where there are only two agents. Based on the exact solution obtained numerically, we present in Figure 1 the parametric plots showing the relation between the cross-sectional dispersion of portfolio holdings \( \text{var}_a[\pi(a)] \) and the conditional volatility of stock returns (scaled by the volatility of endowment process). There are two plots, each for a different degree of heterogeneity across the two investors, which is given by the parameter \( \epsilon \). The solid line in each plot corresponds to the asymptotic solution while the dashed line is for the exact numerical solution. Consistent with our analytic asymptotic results, both plots show that the volatility of stock returns tends to increase with the cross-sectional dispersion of portfolio holdings. Given the nature of the asymptotic expansions we are using, it is not surprising to find that the asymptotic solution is closer to the exact solution in the first plot, where the degree of heterogeneity in the economy is smaller (\( \epsilon \) closer to zero) and investors have risk aversion closer to unity.

Figure 2 compares the asymptotic and numerical solutions for the constrained economy to those in its unconstrained counterpart. The asymptotic solution captures the qualitative features of the exact numerical solution, the quality of approximation being higher in case of the equity premium. The top panels illustrate that imposing leverage constraints lowers the conditional volatility of stock returns, while the bottom panels show that the equity premium rises when leverage constraints are imposed.

4 Conclusion

In this article, we study a general equilibrium exchange economy with multiple agents who differ in their degree of risk aversion and face borrowing constraints. We characterize the consumption
and portfolio policies and also the properties of asset returns using asymptotic analytical methods and numerical methods. We find that the volatility of stock returns increases with the cross-sectional dispersion of risk aversion, and with the relaxation of the constraint on borrowing, and in equilibrium, this is associated with an increase in the cross-sectional dispersion in portfolio holdings. Moreover, tightening the borrowing constraint lowers the risk-free interest rate and raises the equity premium.
Appendix: Proofs for all propositions

Proof of Proposition 1

The result follows by substituting (11) into (7). First- and higher-order terms in the expansion of $g(X)$ do not affect the first-order asymptotic expansion of the optimal consumption-portfolio policy. The equivalent asymptotic expression (14) is obtained by expanding (13) in powers of $\epsilon$ and eliminating terms of order two and higher.

To obtain $g_0$, we use the definition of the value function of the log-utility maximizer, with optimal consumption $c(X_t, t) = \rho$ substituted in,

$$J(W_t, X_t) = E_t \left[ \int_t^{\infty} e^{-\rho(s-t)} \ln (\rho W_s) \, ds \right], \quad (A1)$$

where the wealth process $W_t$ evolves according to

$$\frac{dW_t}{W_t} = \left( -\rho + r(X_t) + \pi(X_t, t) \left( \mu_P(X_t) - r(X_t) \right) \right) \, dt + \pi(X_t, t) \sigma_P(X_t) \, dZ_t.$$

Thus,

$$\ln(W_s) = \ln(W_t) + \int_t^s -\rho + r(X_u) + \frac{\phi(X_u)^2 \sigma_P(X_u)^2}{2} \, du + \int_t^s \phi(X_u) \sigma_P(X_u) \, dZ_u,$$

where we have used the expression for the optimal portfolio policy of the log agent, $\pi(X_t) = \phi(X_t)$. Substituting this into (A1) yields

$$J(W_t, X_t) = E_t \left[ \int_t^{\infty} e^{-\rho(s-t)} \left( \ln \rho + \left( \ln(W_t) + \int_t^s -\rho + r(X_u) + \frac{\phi(X_u)^2 \sigma_P(X_u)^2}{2} \, du \right) \right) \, ds \right].$$

Integration by parts completes the proof of the proposition.

Proof of Proposition 2

As in Proposition 1, this result follows by replacing the function $g(X)$ in the expression for the optimal consumption-portfolio policy with its asymptotic expansion. Only the leading term in the expansion must be retained, which corresponds to the solution of the log-utility maximizer’s
problem. To obtain $g^c$ note that the wealth process of the log-investor evolves according

$$\frac{dW_t}{W_t} = \left(-\rho + r(X_t) + \pi_0(X_t)\phi(X_t)\sigma_p(X_t)^2\right) dt + \pi_0(X_t)\sigma_p(X_t) dP_t,$$

where $\pi_0(X_t)$ is the optimal portfolio policy of the log-utility maximizer, given by (19). Repeating the steps of the proof of Proposition 1 we obtain the desired result.

**Proof of Proposition 3**

Substituting the expansions (21) into the expression for the consumption policy and portfolio weights in Proposition 1, and eliminating the higher-order terms, yields the result.

**Proof of Proposition 4**

Divide equation (10) by aggregate wealth, which equals $P_t$, to get

$$\frac{e_t}{P_t} = \sum_a c_t(a)\omega_t(a).$$

From equation (23), the individual consumption policy is

$$c_t(a) = \rho - \epsilon a\rho A + O(\epsilon^2).$$

Together, these two results imply that

$$\frac{P_t}{e_t} = \frac{1}{\rho} \sum_a \frac{1}{1 - \epsilon aA} \omega_t(a) + O(\epsilon^2) = \frac{1}{\rho} (1 + \epsilon A\mathbb{E}_a[a]) + O(\epsilon^2). \quad (A2)$$

Using

$$\frac{dP_t}{P_t} = \frac{d(P_t/e_t)}{(P_t/e_t)} + \frac{de_t}{e_t} + \frac{de_t}{e_t} \cdot \frac{d(P_t/e_t)}{P_t/e_t}, \quad (A3)$$

and

$$d\omega_t(a) = O(\epsilon) dt + O(\epsilon) dZ_t \quad \text{(A4)}$$

we find that

$$\frac{dP_t}{P_t} = \mu_e dt + \sigma_e dZ_t + O(\epsilon^2),$$
which then implies that

\[ \mu_{Rt} = \mu_e + \frac{\epsilon_t}{R_t} + O(\epsilon^2) \]

\[ = (\mu_e + \rho) - \epsilon \rho A E_a[a] + O(\epsilon^2) \]

and

\[ \sigma_{Rt} = \sigma_e + O(\epsilon^2). \]

Using the condition in equation (9) for equilibrium in the stock market,

\[ \frac{\mu_{Rt} - r_t}{\sigma_{Rt}^2} (1 + \epsilon E_a[a]) + O(\epsilon^2) = 1, \]

we have that

\[ \frac{\mu_{Rt} - r_t}{\sigma_{Rt}^2} = 1 - \epsilon E_a[a] + O(\epsilon^2) \quad (A5) \]

so that

\[ r_t = \mu_{Rt} - \sigma_{Rt}^2 (1 - \epsilon E_a[a]) + O(\epsilon^2), \]

\[ = (\mu_e - \sigma_e^2 + \rho) + \epsilon(\sigma_e^2 - \rho A)E_a[a] + O(\epsilon^2). \]

Also, using (A5), the expression for the optimal portfolio weight in equation (24) reduces to

\[ \pi_t(a) = 1 + \epsilon (a - E_a[a]) + O(\epsilon^2). \]

To derive the process for stock returns in terms of exogenous variables, and to determine the higher-order terms in the asymptotic expansions of the moments of the return process, we start by describing the evolution of \( W_t(a) \):

\[
\frac{dW_t(a)}{W_t(a)} = \left[ \pi_t(a)(\mu_{Rt} - r_t) + r_t - c_t(a) \right] dt + \pi_t(a)\sigma_{Rt} dZ_t \\
= \left[ \sigma_e^2(1 - \epsilon E_a[a]) \left( 1 + \epsilon(a - E_a[a]) \right) + \mu_e - \sigma_e^2 + \rho + \epsilon(\sigma_e^2 - \rho A)E_a[a] - \rho(1 - \epsilon a A) \right] dt \\
+ \sigma_e \left( 1 + \epsilon(a - E_a[a]) \right) dZ_t + O(\epsilon^2)
\]
\[
\sigma_e^2 + \epsilon \sigma_e^2 (a - 2E_a[a]) + \mu_e - \sigma_e^2 + \rho + \epsilon (\sigma_e^2 - \rho A)E_a[a] - \rho + \epsilon \rho Aa \ dt
+ \sigma_e (1 + \epsilon (a - E_a[a])) \ dZ_t + O(\epsilon^2)
\]
\[
\mu_e + \epsilon (\sigma_e^2 + \rho A)(a - E_a[a]) \ dt + \sigma_e (1 + \epsilon (a - E_a[a])) \ dZ_t + O(\epsilon^2).
\]

Next,
\[
\frac{d\omega_t(a)}{\omega_t(a)} = \frac{dW_t(a)}{W_t(a)} - \frac{dP_t}{P_t} + \frac{1}{P_t^2} \left[ dP_t, dP_t \right] - \left[ \frac{dW_t(a)}{W_t(a)}, \frac{dP_t}{P_t} \right]
\]
\[
= \left[ \mu_e + \epsilon (\sigma_e^2 + \rho A)(a - E_a[a]) \right] dt + \sigma_e(1 + \epsilon (a - E_a[a]))dZ_t - \mu_e dt - \sigma_e dZ_t
+ \sigma_e^2 dt - \sigma_e^2 (1 + \epsilon (a - E_a[a])) dt + O(\epsilon^2)
\]
\[
= \epsilon \rho A(a - E_a[a]) dt + \epsilon \sigma_e(a - E_a[a])dZ_t + O(\epsilon^2).
\]

Finally, this leads to the following result: from (A2),
\[
d(P_t/e_t) = \left( \frac{\epsilon A}{\rho} \right) \left( \sum_a a \ d\omega_t(a) \right) + O(\epsilon^2) \cdot d\omega_t(a)
\]
\[
= \epsilon^2 \left( A^2 \sum_a a (a - E_a[a]) \omega_t(a) \right) dt + \epsilon^2 \frac{A \sigma_e}{\rho} \sum_a a (a - E_a[a]) \omega_t(a) dZ_t + O(\epsilon^3)
\]
\[
= \epsilon^2 A^2 \text{var}_a[a] dt + \epsilon^2 A \sigma_e \frac{\text{var}_a[a]}{\rho} dZ_t + O(\epsilon^3),
\]
and so, from (A3),
\[
\frac{dP_t}{P_t} = \epsilon^2 \rho A^2 \text{var}_a[a] dt + \epsilon^2 A \sigma_e \text{var}_a[a] dZ_t + \mu_e dt + \sigma_e dZ_t + \epsilon^2 A \sigma_e^2 \text{var}_a[a] dt + O(\epsilon^3)
\]
\[
= (\mu_e + \epsilon^2 (\rho A^2 + \sigma_e^2 A) \text{var}_a[a]) dt + (\sigma_e + \epsilon^2 A \sigma_e \text{var}_a[a]) dZ_t + O(\epsilon^3).
\]

**Proof of Proposition 5**

The result follows from differentiating the expression for the volatility of stock returns in (27) with respect to \( \text{var}_a[a] \).
Proof of Proposition 6

The introduction of portfolio constraints changes equation (28) and onwards, but because equations (25) and (26) are based on (23), these results are still valid.

From equation (9):

\[ 1 = E_a \left[ \pi_t(a) \right] \]

\[ = E_a \left[ (1 + \epsilon a) \frac{\mu_R - r_t}{\sigma_R^2} \chi \left( a \leq L \frac{\sigma_R^2}{\mu_R - r_t} \right) \right] + E_a \left[ (1 + \epsilon L) \chi \left( a > L \frac{\sigma_R^2}{\mu_R - r_t} \right) \right], \]

where \( \chi(\cdot) \) is an indicator function. This equation is used to determine the interest rate, \( r_t \), as follows. Assuming that \( a' < a'' \) and that the constraint binds for \( a'' \),

\[ 1 = \left( 1 + \epsilon a' \right) \frac{\mu_R - r_t}{\sigma_R^2} \omega_t(a') + (1 + \epsilon L) \omega_t(a''), \]

which implies that

\[ \frac{\mu_R - r_t}{\sigma_R^2} = \frac{\omega_t(a') - \epsilon L \omega_t(a'')}{\omega_t(a') (1 + \epsilon a')} \]

\[ = 1 - \epsilon \left( L \frac{\omega_t(a'')}{\omega_t(a')} + a' \right) + O(\epsilon^2) \]

\[ r_t = \mu_R - \sigma_R^2 \left[ 1 - \epsilon \left( L \frac{\omega_t(a'')}{\omega_t(a')} + a' \right) \right] + O(\epsilon^2). \quad (A6) \]

According to (32), the portfolio policies are then given by

\[ \pi_t(a) = 1 + \epsilon \min \left[ a - \left( L \frac{\omega_t(a'')}{\omega_t(a')} + a' \right), L \right] + O(\epsilon^2). \]

Consider the region where the constraint is binding, i.e., where \( \pi_t(a'') = 1 + \epsilon L \):

\[ L \leq a'' - L \frac{\omega_t(a'')}{\omega_t(a')} - a' \]

\[ \Rightarrow \omega_t(a') \geq \frac{L}{a'' - a'}. \]
In this region,

\[
\frac{dW_t(a')}{W_t(a')} = \left[ \mu_e + \epsilon \left( \rho A (a' - E_a[a]) - \sigma^2_e L \frac{\omega_t(a'')}{\omega_t(a')} \right) \right] dt \\
+ \sigma_e \left( 1 + \epsilon a' \right) \left[ 1 - \epsilon \left( L \frac{\omega_t(a'')}{\omega_t(a')} + a' \right) \right] dZ_t \\
= \left[ \mu_e + \epsilon \left( \rho A (a' - E_a[a]) + L \sigma^2_e L \frac{\omega_t(a'')}{\omega_t(a')} \right) \right] dt \\
+ \sigma_e \left[ 1 - \epsilon L \frac{\omega_t(a'')}{\omega_t(a')} \right] dZ_t + O(\epsilon^2);
\]

\[
\frac{dW_t(a'')}{W_t(a'')} = \mu_e + \epsilon \left( \rho A (a'' - E_a[a]) + L \right) dt + \sigma_e (1 + \epsilon L) dZ_t + O(\epsilon^2).
\]

Thus,

\[
\frac{d\omega_t(a')}{\omega_t(a')} = \epsilon \rho A \left( a' - E_a[a] \right) \ dt - \epsilon \sigma_e L \frac{\omega_t(a'')}{\omega_t(a')} \ dZ_t;
\]

\[
\frac{d\omega_t(a'')}{\omega_t(a'')} = \epsilon \rho A \left( a'' - E_a[a] \right) \ dt + \epsilon \sigma_e L \ dZ_t.
\]

Using these results,

\[
d \left( \frac{P_t}{v_t} \right) = \left( \frac{\epsilon A}{\rho} \ dZ_t + O(\epsilon^2) \right) \left( \sum_a a \ d\omega_t(a) \right)
\]

\[
= \left[ \ldots \right] dt + \epsilon^2 \frac{A \sigma_e L}{\rho} \left[ a'' \omega_t(a'') - a' \omega_t(a'') \right] dZ_t + O(\epsilon^3)
\]

\[
= \left[ \ldots \right] dt + \epsilon^2 A \sigma_e L \left[ a'' - a' \right] \omega_t(a'') \ dZ_t + O(\epsilon^3).
\]

So,

\[
\sigma_{Rt} = \sigma_e + \epsilon^2 A \sigma_e L \left[ a'' - a' \right] \omega_t(a'') + O(\epsilon^3). \tag{A7}
\]

Substituting this expression into (A6) yields the formula for the risk-free rate.
Proof of Proposition 7

Using the expression in (A7) and the fact that the leverage constraint binds only when
\[ \omega_t(a')(a'' - a') \geq L, \]
we see that
\[ \sigma^2_{R_t} \leq \sigma^2_e + \epsilon^2 A \sigma_e \left[ a'' - a' \right]^2 \omega_t(a') \omega_t(a'') + O(\epsilon^3). \]

Note that
\[
\text{var}[a] = (a')^2 \omega_t(a') + (a'')^2 \omega_t(a'') - \left[ a' \omega_t(a') + a'' \omega_t(a'') \right]^2 \\
= (a')^2 \omega_t(a') \omega_t(a'') + (a'')^2 \omega_t(a') \omega_t(a'') - 2a'a'' \omega_t(a') \omega_t(a'') \\
= (a'' - a')^2 \omega_t(a') \omega_t(a'').
\]

Therefore,
\[ \sigma^2_{R_t} \leq \sigma^2_u + O(\epsilon^3). \]

Proof of Proposition 9

Assume that agents have risk aversion between \(-\hat{a}\) to \(+\hat{a}\), with a general wealth distribution, \(\omega(a)\).

Also,
\[ \phi_t = \phi_0 + \epsilon \phi_{1t} + O(\epsilon^2) = 1 + \epsilon \phi_{1t} + O(\epsilon^2), \]
where the leading term in the expansion \(\phi_0 = 1\) is the market price of risk in the economy populated with log-utility agents. Then, the stock-market-clearing condition in equation (33) implies that:
\[ 0 = \int_{-\hat{a}}^{L-\phi_{1t}} \omega_t(a) da + \int_{L-\phi_{1t}}^{\hat{a}} L \omega_t(a) da. \]

Differentiating the above with respect to \(L\), we have:
\[ 0 = \int_{-\hat{a}}^{L-\phi_{1t}} \frac{\partial \phi_{1t}}{\partial L} \omega_t(a) da + (L - \phi_{1t}) \omega_t(L - \phi_{1t}) \left( 1 - \frac{\partial \phi_{1t}}{\partial L} \right) \\
- L \omega_t(L - \phi_{1t}) \left( 1 - \frac{\partial \phi_{1t}}{\partial L} \right) + \int_{L-\phi_{1t}}^{\hat{a}} \omega_t(a) da, \]
which then implies that
\[ \frac{\partial \phi_{1t}}{\partial L} = -\frac{\int_{L-\phi_{1t}}^{\hat{a}} \omega_t(a) da}{\int_{-\hat{a}}^{L-\phi_{1t}} \omega_t(a) da} \leq 0. \quad (A8) \]
Since the first-order terms in expansions of the mean and the volatility of stock returns are not affected by the leverage constraint, this implies that asymptotically the interest rate in the constrained economy is lowered and the equity premium is increased by tightening the borrowing constraint.

Now, we compute the volatility of the returns process:

\[
d \left( \frac{P_t}{e_t} \right) = \frac{A}{\rho} \int_{-\hat{a}}^{\hat{a}} a \omega_t(a) \, da + O(\epsilon^3).
\]

The wealth of agents for whom the constraint does not bind changes according to

\[
\frac{dW_t(a)}{W_t(a)} = [...] \, dt + [1 + \epsilon (a + \phi_1t)] \sigma_e dZ_t + O(\epsilon^2),
\]

\[
\frac{d\omega_t(a)}{\omega_t(a)} = [...] \, dt + \epsilon (a + \phi_1t) \sigma_e dZ_t + O(\epsilon^2).
\]

For the constrained agents,

\[
\frac{d\omega_t(a)}{\omega_t(a)} = [...] \, dt + \epsilon L \sigma_e dZ_t + O(\epsilon^2),
\]

so,

\[
d \left( \frac{P_t}{e_t} \right) = [...] \, dt + \epsilon^2 A \sigma_e \rho \left[ \int_{-\hat{a}}^{L-\phi_{1t}} a \omega_t(a) (a + \phi_{1t}) \, da + \int_{L-\phi_{1t}}^{\hat{a}} L a \omega_t(a) \, da \right] dZ_t + O(\epsilon^3).
\]

Thus,

\[
\frac{\partial \sigma_{Rt,2}}{\partial L} = \epsilon^2 A \sigma_e \rho \left[ \int_{-\hat{a}}^{L-\phi_{1t}} a \omega_t(a) \frac{\partial \phi_{1t}}{\partial L} \, da + \int_{L-\phi_{1t}}^{\hat{a}} a \omega_t(a) \, da \right]
\]

\[
= \epsilon^2 A \sigma_e \rho \left[ \int_{-\hat{a}}^{L-\phi_{1t}} (a - L + \phi_{1t}) \omega_t(a) \frac{\partial \phi_{1t}}{\partial L} \, da + \int_{L-\phi_{1t}}^{\hat{a}} (a - L + \phi_{1t}) \omega_t(a) \, da \right]
\]

\[
\geq 0
\]

where \( \sigma_{Rt} = \sigma_e^2 + \epsilon^2 \sigma_{Rt,2} + O(\epsilon^3) \) and we have used (A8) to establish the second equality and to determine the sign of \( \partial \phi_{1t} / \partial L \).
Figure 1: Volatility of stock returns in a heterogeneous economy

The ratio of the conditional volatility of stock returns to the volatility of the endowment process, $\sigma_{Rt}/\sigma_e$, is plotted against the cross-sectional variance of portfolio holdings, $\text{var}_a[\pi(a)]$. The solid line corresponds to the analytic asymptotic solution, $\sigma_{Rt} = \sigma_e + A \sigma_e \text{var}_a[\pi(a)]$; the dashed line is computed numerically. The following parameter values are used: $\mu_e = .02$, $\sigma_e = .03$, $\rho = .02$. There are two types of agents in the economy, $a' = 0$ and $a'' = -1$. The small parameter $\epsilon$ takes values of .25 and .5.
Figure 2: Volatility and equity premium in an economy with constraints on leverage

The panels on the left plot the ratio of the conditional volatility of stock returns to the volatility of the endowment process, $\sigma_R/\sigma_e$. The panels on the right plot the ratio of the conditional equity premium to the squared volatility of the endowment process, $(\mu_R - r_t)/\sigma^2_e$. In both cases, the argument is the cross-sectional distribution of wealth, $\omega(a'')$. The solid line corresponds to the analytic asymptotic solution in the unconstrained economy, the dashed line is computed numerically for the unconstrained economy. Triangles mark the asymptotic solution in the constrained economy, squares mark the numerical solution in the constrained economy. The following parameter values are used: $\mu_e = .02, \sigma_e = .03, \rho = .02$. The leverage constraint is given by $L = .5$. There are two types of agents in the economy, $a' = 0$ and $a'' = -1$. The small parameter $\epsilon$ takes values of .25 and .5.
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