Correlation functions for a one-dimensional Fermi system with long-range interaction (Tonomaga model)

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Communicated March 1, 1973


The correlation functions (Green functions) are found for a one-dimensional system of Fermi particles with long-range interaction (Tonomaga model). It is shown that such a system does not behave like Fermi liquid since the Green function does not possess a pole and a "Fermi step" is absent from the momentum distribution of particles.

1. INTRODUCTION

In recent years there has been intensive study of conductors (or semiconductors) of a special type, in which the electrons execute an essentially one-dimensional motion. An example is organic materials containing TCNQ. In these materials the molecules form threads separated by such large distances that practically no jumps of electrons occur from one thread to another. The structure of the electronic spectrum in such quasi-unidimensional systems differs significantly from the spectrum of a three-dimensional system, especially in the case of a "quasiunidimensional metal": that is, when the number of electrons associated with a single molecule of the chain is less than two. Therefore it is especially important to investigate exactly solvable models of a one-dimensional system.

There are two groups of such models: the Fermi-gas model with \( \delta \)-function interaction, which was investigated in the papers of Gorkin, of Yang and of Lieb and Wu [1], and the Fermi-gas model with long-range interaction, first considered by Tonomaga [2] and in a somewhat expanded form by Lieb and Mattis [3]. (Both these models are reproduced also in the book of Lieb and Mattis [4]).

The exact solutions made it possible to calculate the energy of the ground state, the energy of the Bose branches of the spectrum, and also some characteristics of the one-electron spectrum, for example, the difference between the chemical potentials corresponding to addition or removal of one particle: \( \mu - \mu_c \). But the exact wave function does not allow to calculate the correlation functions or to solve the problem of the presence or absence of long-range order (in the ground state; at finite temperatures, of course, there is no long-range order) by means of them is practically impossible. The only exception so far is Lieb and Mattis' [5] calculation of the distribution \( n(q) \) of particles with respect to momentum.

At the same time, the correlation functions and the long-range order parameters \( \gamma_r \) have been calculated \( \delta \)-function interaction, and in our previous paper [6] by perturbation theory, within the framework of the so-called logarithmic approximation. Although these correlation functions are not only in qualitative, but also with exponential accuracy in quantitative agreement with the results of the exact theory [7], nevertheless the possibility of extending them to the case of strong interaction seems very doubtful.

The fact is that such an expansion implies retention in the one-dimensional Fermi system of all the properties of a Landau Fermi liquid, and most importantly of the pole characteristics of the single-time Green function [2]:

\[
G(\omega = -i \omega) = 1 + \frac{\beta}{\omega_p - \omega} \geq
\]

where \( \gamma_r \) is the velocity on the "Fermi surface", which is one-dimensional in the case of two points \( \gamma_r = \gamma_p \) (see Fig. 1); \( \delta < 0 \), \( \delta = 0 \) is a constant that determines, as is well known (see [2]), the jump of the particle distribution function \( n(q) \) at the Fermi surface \( \gamma_r = \gamma_p \).

Consideration of the problem by perturbation theory has already shown that formula (1) certainly does not hold in the one-dimensional case with the highest (nonlogarithmic) accuracy (for details see Sec. 7). It is therefore important to find \( G(p) \) in one of the exactly solvable models mentioned above, and to compare the result with formula (1).

It has been shown that such a calculation can be carried out in the Tonomaga model. The Tonomaga model is a gas of Fermi particles with density \( \rho \) (Fermi momentum \( p = \pi / 2 \)), mass \( m \) (velocity \( v = \rho / m \)), and interaction potential \( \lambda(q) \), whose Fourier components \( \lambda_k \) are different from zero only in an extremely narrow momentum interval \( |k| \ll \rho \). The corresponding Hamiltonian has the form

\[
H = \sum_{\langle i,j \rangle} \lambda_q \phi_i \phi_j \geq
\]

where \( \phi_q \) is the wave function for the state \( \langle q \rangle \), and \( \lambda_q \) is the interaction potential.

Tonomaga showed that in the case of extremely long-range interaction, that is with neglect of all quantities of the type \( \lambda_{pq} / \rho^2 \), the spectrum of the Hamiltonian is coincident with the spectrum of a system of bosons \( b_q \) with the Hamiltonian

\[
H = \sum_q \epsilon_q b_q^\dagger b_q \geq
\]

as is shown below, the Green function retains the form (1) in the region \( |p| \ll \rho \) but has a quite different form near the Fermi surface, \( |p| \gg \rho \) is a region with a new kind of correlation function which is \( \gamma_r \) is a parameter of the spectrum.

2. BASIC EQUATIONS: WARD’S IDENTITY

We shall calculate the Green function near the right and left "Fermi points": \( G(p) \) and \( G(p) \) (see Fig. 1). For the free gas

\[
G(q) = \frac{1}{\epsilon_q + \Pi(q)} \geq
\]

where, as above, in the arguments of the functions we denote by \( k \) the pair \( (q, \omega) \).

The relations (8) can be derived directly by considering the diagrams of different orders and by using the identity, which follows from (4),

\[
G(p_1, \omega_1) G(p_2, \omega_2) = G(p_1, \omega_1 + \omega_2) G(p_2, \omega_2 + \omega_1) \geq
\]

where \( \Pi(q) \) is the interaction potential (see the text).

Here it is necessary to remember the following two facts: first, since the interaction (for broken line diagrams) transfers only a practically zero momentum \( \Lambda \), there is in all diagrams for \( G \) a continuous line of a particle
In these regions, with accuracy $\kappa - A > 0$, we may suppose that $n_{\parallel}(p) = \eta(p)$, where $\eta$ is the Fermi step at $T = 0$. Therefore

$$\nu_{s} \eta(\nu_{s} - \kappa) = \eta(\nu_{s} - \kappa).$$

(14)

Analogous calculations for $\Pi_{g}$ give

$$\nu_{s} \eta(\nu_{s} - \kappa) = \eta(\nu_{s} - \kappa).$$

(15)

Now solving equation (5), we find

$$D_{\parallel}(\lambda) = \frac{\nu_{s} \eta(\nu_{s} - \kappa) - \nu_{s} \eta(\nu_{s} - \kappa)}{\nu_{s} \eta(\nu_{s} - \kappa)}.$$

(16)

Here and elsewhere we omit the index $G$ on $G$, and measure $\kappa$ from $p_{0}$ ($\kappa < \nu_{s} - \kappa$). It is easily shown by direct substitution that equation (19) is satisfied by the function

$$G(\nu_{s} - \kappa) = \frac{\nu_{s} \eta(\nu_{s} - \kappa)}{\nu_{s} \eta(\nu_{s} - \kappa)}.$$

(20)

and for G we have the equation

$$(\nu_{s} - \kappa) G(\nu_{s} - \kappa) = \frac{\nu_{s} \eta(\nu_{s} - \kappa)}{\nu_{s} \eta(\nu_{s} - \kappa)}.$$

(21)

We proceed to the case of small $\lambda_{1}$ and $\lambda_{2}$: that is, $u(\nu_{s})$ is close to unity. We express the integral co

$$\nu_{s} \eta(\nu_{s} - \kappa) = \frac{\nu_{s} \eta(\nu_{s} - \kappa)}{\nu_{s} \eta(\nu_{s} - \kappa)}.$$

(22)

We shall be concerned with the momentum region, $\nu_{s} \approx \kappa < \kappa$. In the range where $\nu_{s} \approx \kappa$, as will be seen, the old answer (20) is retained with a value $\nu_{s}$ instead of $\nu_{s}$. By iteration of equation (22), it can be shown that the integration in the first integral extends over a narrow region $u = \kappa \approx \nu_{s} - \kappa$. On the contrary, in the second term

$$-\lambda_{1} \frac{\nu_{s} \eta(\nu_{s} - \kappa)}{\nu_{s} \eta(\nu_{s} - \kappa)}.$$

(23)

The result of the integration over the momentum $\nu_{s}$ depends, of course, on the specific law of approach of $\nu_{s}$ to zero for $\nu_{s} \approx \kappa$. Formula (23) was obtained for the simplest form of such cutoff, we considered $\nu_{s}$ and $\kappa$ constant, but introduced into the integral a cutoff multiplier $\nu_{s} \approx \kappa$. Therefore the results obtained are quantitatively applicable only in the regions $x, t > 1/\Lambda$ and $x, t < 1/\Lambda$.}

Equation (27) is to be solved under two boundary conditions. The first condition is that, by definition,

$$G_{\nu_{s} = \nu_{s} + \kappa}(0) = G_{\nu_{s} = \nu_{s} + \kappa}(0).$$

(24)

The second condition is the well-known analyticity property of $G$ as a function of complex $\nu_{s}$ (see, for example, (25, Sec. 7). In the coordinate representation, this condition reads: for $t > 0$, the function $G(t)$ can be continued analytically from the real axis into the right lower quadrant of the complex variable $t$ ($t > 0$, $|t| < \kappa$); for $t < 0$, on the other hand, what is analytic is the continuation of $G(t)$ from the left semicircle into the left upper quadrant ($t < 0$, $|t| > \kappa$).

The Green function $G(\nu_{s}, t)$ of free particles, which of course satisfies both boundary conditions, has the form

$$G(\nu_{s}, t) = \frac{1}{i\pi} \left[ \delta(t) - \frac{1}{2} \delta(t - \nu_{s}^{2}) \right].$$

(25)

For example, its solution for $t > 0$ satisfying condition (29) can be written in the form

$$G_{\nu_{s} = \nu_{s} + \kappa}(0) = G_{\nu_{s} = \nu_{s} + \kappa}(0).$$

(30)

Thus for small $\lambda_{1}$ and $\lambda_{2}$,

$$G_{\nu_{s} = \nu_{s} + \kappa}(0) = G_{\nu_{s} = \nu_{s} + \kappa}(0).$$

(31)

A formula of this type was first obtained by Lieb and Mattis (26).

4. THE CASE $\lambda_{1} = \lambda_{2} = \lambda$.

Interactions of arbitrary intensity we shall consider only for the physically realistic case

$$\Delta_{\parallel}(\nu_{s}) = \lambda_{1} \lambda_{2} = \lambda(\nu_{s}).$$

(26)

In (26) we perform the inverse transformation to ordinary coordinate space $u, t$. Then for $t = 0$

$$G(\nu_{s}, t) = \frac{1}{i\pi} \left[ \delta(t) - \frac{1}{2} \delta(t - \nu_{s}^{2}) \right].$$

(32)

The analogous calculations for $t < 0$ give a formula that differs from (32) only by the substitution $A \rightarrow A$ and $t \rightarrow -t$. Condition (29), however, requires that

$$\lambda_{1} f(t) = \lambda_{1} f(t).$$

(33)

In formula (32) and the corresponding formula for $t < 0$, only one factor has incorrect analytic behavior, namely

$$\lambda_{1} f(t) = \lambda_{1} f(t).$$

(34)

By use of standard theorems on analytic continuation,

$$\lambda_{1} f(t) = \lambda_{1} f(t).$$

(35)

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The final formula for \( G_\lambda \), the function for which we are trying to show that this choice is unique, is

\[
G_\lambda(t) = \frac{1}{2\pi} \int_0^{\infty} e^{-z^2} \left( \frac{z}{\alpha} \right)^{\lambda - 1} \frac{1}{z} \, dz,
\]

where \( \alpha = \sqrt{\lambda} \), and \( z = t^2 \).

For weak interaction, when \( \lambda = 0 \), \( \nu = 1 \), and \( \alpha = 0 \), formula (35) gives a result that coincides with (25) for \( \lambda = \lambda_0 = \lambda \). Formula (25) for the singularity of \( \nu \) near the “Fermi point” \( p = 0 \) remains valid as long as \( \alpha < \sqrt{\lambda/\nu} \); that is, as long as the interaction is not too strong:

\[
s_n = \frac{1}{\pi} \text{const} \cdot \left| \mathbf{p} \right|^2 \eta (\mathbf{p}) \eta (\mathbf{p})(\mathbf{p}) d\mathbf{p}.
\]

With further increase of the strength of the interaction, as \( \alpha \) becomes greater than \( \sqrt{\nu} \), the leading term in the expansion of \( \nu \) near the Fermi point becomes a linear term:

\[
s_n = \frac{1}{\pi} \text{const} \cdot \left| \mathbf{p} \right|^2 \eta (\mathbf{p}) \eta (\mathbf{p})(\mathbf{p}) d\mathbf{p}.
\]

We have not succeeded in calculating the Fourier components of \( G_\lambda \) in explicit form. It is possible, however, to obtain easily the behavior of \( G_\lambda \) for \( p < \pi \Delta \). For \( \gamma > 0 \), the value

\[
G_\lambda(p) = \int_0^{\infty} G_\lambda(t) e^{-\pi p t} dt
\]

is determined by the region \( \lambda \gg 1/\Delta \). An estimate by dimensional methods gives directly

\[
G_\lambda(p) \sim \frac{1}{\Delta^1} e^{-\pi p \Delta T}.
\]

For \( \gamma > 0 \), the principal contribution to the integral (37) will come from the region \( \lambda \gg 1/\Delta \), and \( G \) will be finite on the Fermi surface \( p = 0 \).

5. FINITE TEMPERATURES

The Green functions of fermions in the Tomonaga model can be found also for finite temperatures. This is so because Ward’s identities (6) and all the arguments involved in their derivation remain valid also in the diagram technique for finite temperatures, which operates

\[
\text{with the imaginary frequencies } \xi_\nu, \xi_\nu = \frac{2n + 1}{2} \pi T,
\]

where \( n \) is an integer (see, for example, I.-V. Chap. III). Now Ward’s identities connect the temperature-dependent Green functions of the fermions \( G_\lambda(p) \), \( n = 0, 1, \) with the corresponding vertex parts \( \Gamma_\lambda(p) \), \( k = 0, 1, \) for all temperatures \( \theta \), \( \gamma = 2\theta T \). For the temperatures \( \theta \), \( \gamma = 2\theta T \), and effective potential \( V \), all the formulas \( (3)-8 \) and \( (13)-17 \) remain valid for substitutions

\[
G \rightarrow G_\lambda, \quad \Theta \rightarrow \Theta_\lambda, \quad \Delta \rightarrow \Delta_\lambda
\]

and

\[
\int_0^{\infty} dt e^{-\theta t} = \frac{1}{\theta} \; \text{const}.
\]

The Green functions of SC fermions have the periodicity property

\[
\Theta_\lambda(\nu + \Theta) = \Theta_\lambda(\nu), \quad \Theta_\lambda(\nu) = \Theta_\lambda(\nu + \Theta).
\]

Hence

\[
\frac{\pi}{c^p}(\nu + \Theta) = \frac{\pi}{c^p} \nu, \quad \Theta_\lambda(\nu) = \Theta_\lambda(\nu + \Theta).
\]

8. QUASI-DIMENSIONAL METAL WITH COULOMB INTERACTION

The results obtained above can be generalized to the case of a quasidimensional metal in which the distance between the conducting channels is sufficiently large so that the electrons cannot jump from one channel to another. It is not permissible, however, to neglect the interaction between electrons on different channels. It is important to note that all the relations that follow from Ward’s identities apply to such a case as well.

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Frequency dependence of the conductivity of one-dimensional systems

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The dependence of the conductivity of one-dimensional systems on the frequency of an external electric field, \( \omega \), is investigated. It is shown that the static conductivity (\( \omega \to 0 \)) is zero. For sufficiently high electric fields the conductivity tends to zero at a rate slower than \( \omega^4 \).

1. The conductivity of one-dimensional systems has been the subject of many investigations. Nonetheless, the dependence of the conductivity on the frequency of the external electric field has not been determined so far. All the arguments advanced to date concerning this dependence were exclusively indirect and based on investigations of the energy spectra of one-dimensional systems. These include the researches by Mott and Twose[11] and Borland[12]. Using purely intuitive considerations (Mott) or investigating the ergodicity properties of the equation for the distribution function of the phase shifts of the particle wave function (Borland), these authors reached the conclusion that the wave function is localized, and indicated by the same token the possibility that one-dimensional systems have static conductivity. No direct calculation of the conductivity was made, however, and the question remained open.

We show here on the basis of exact equations for the conductivity that it is equal to zero at \( \omega = 0 \) and that it decreases with frequency at a rate not lower than \( \omega^{-4} \) at sufficiently high frequencies.

The first exact equations for the conductivity were derived by Halperin[13] for the particular case of white noise. Dynnikov[14] investigated the problem of averaging the product of two Green functions of a particle. The present paper deals with the case when the impurity potential is of the form \( U(x) = U_0 \delta(x - x_0) \) and the distances between impurities have a Poisson distribution.

The plan of the paper is the following. In Sec. 2 we introduce the characteristic functions \( \Phi(a, a') \) and the equations they satisfy [Eqs. (12)] are derived. The conductivity \( \sigma \) is expressed in terms of these functions with the aid of formulas (1) and (10). We note here that the quantity \( \mid \Phi(a, a') \mid \) is replaced in (1) by the derivative \( \delta \Phi/\delta a \). Equations (12) are investigated in Sec. (12) in the case of high energies, when the collision term in (12) takes the form of (13). It is shown that in this approximation the conductivity decreases with frequency so slowly that \( \sigma \to \omega \), and the limiting expression for the function \( \Phi(a, a') \) takes the form of a \( \delta \) function, i.e., \( \Phi(a, a') \sim \delta(a - a') \) as \( \omega \to 0 \). It is shown in Sec. 7 that \( \Phi(a, a') = (a - a') \) is an exact solution of (12) at all values of the energy \( \omega \). From this and from formulas (10) and (1) it follows that there is no static conductivity at any energy.

2. We start with the well known expression for the conductivity in an external electric field of frequency \( \omega \) (we put \( \Sigma = 1 \) throughout):

\[
\sigma(\omega) = -\frac{\omega}{2} \int \left[ \Phi_{\omega} \Phi_{\omega}^* \right] \, d\Phi_{\omega} 
\]

where \( \Phi_{\omega} \) is the particle energy, \( \epsilon(\omega) \) is the particle energy distribution function, and

\[
\Phi_{\omega}(a) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{a^2}{2x^2}} \, dx
\]

In the one-dimensional case, formula (2) can be represented in a different form by introducing the logarithmic derivative of the wave function of the electron, \( s(\omega) = \Phi_{\omega}^{\prime}(a) / \Phi_{\omega}(a) \), which satisfies the following equation (k = 2m).

\[
s(\omega) = -i(\omega^2 + k^2)^n \left[ \delta_{\text{Poisson}} - \delta_{\text{f}} \right]
\]

The energy levels of a system with dimensions \( L \) are determined by the boundary conditions (the energy is a parameter in Eq. (9)).

It is now easy to rewrite the expression for \( \Phi(a, a') \) in the form

\[
\Phi_{\omega}(a, a') = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{a^2}{2x^2}} \, dx
\]

The angle brackets denote averaging over the impurity coordinates \( \xi \). From (9) we find that

\[
\Phi_{\omega}(a, a') \propto e^{-2a\xi \xi'} \epsilon(\omega) \epsilon(\omega - 1)
\]

and finally we have for \( \Phi(a, a) \)

\[
\Phi(a, a) \propto e^{-a^2 \xi \xi'} \epsilon(\omega) \epsilon(\omega + 1)
\]

We note here that in the derivation of the expression for \( \Phi(a, a) \) in (4) we have used the definition (2) of \( \Phi \), and then have written down in explicit form the square of the matrix element of the current and have taken into account the fact that...