cross section by a function $\sigma(p^2)$, then in the energy region below 650 Mev, we have $\sigma(p^2) \sim (0.3 \pm 0.1) \times 10^{30}$ cm$^2$.

**Partial Cross Sections**

Within the bounds of the charge-independence hypothesis all the cross sections for pion production in nuclear collisions can be represented as a linear combination of three partial cross sections $\sigma_1$, $\sigma_2$, and $\sigma_3$ (the notation of Ref. 25 is used here; the indices denote the magnitudes of the isospin spins of the nucleons in the initial and final states). Using our data and that of Refs. 27 and 31, it is possible to determine the magnitudes of these cross sections at the proton energy 660 Mev:

\[ \sigma_1 = (3.6 \pm 0.2) \times 10^{31} \text{ cm}^2, \]
\[ \sigma_2 = (0.6 \pm 0.8) \times 10^{31} \text{ cm}^2, \]
\[ \sigma_3 = (10.6 \pm 0.6) \times 10^{31} \text{ cm}^2. \]

The cross section $\sigma_1$ is $\sigma(p^2)$ at various energies has been determined above (see Fig. 7). A comparison of the magnitudes of the cross section $\sigma_n$ at various energies indicates (Fig. 11) that $\sigma_n$ varies as $p_n^2$, which is in good agreement with the conclusions of Rosenfel'd.

The reaction \( p + n \rightarrow n + p \) is characterized by a slower rate of increase of the cross section with energy (see Fig. 11); in the region up to 580 Mev, the cross section $\sigma_n$ is proportional to the third power of the momentum:

\[ \sigma_n = (2.0 \pm 0.4) \times 10^{-4} \times 10^{27} \text{ cm}^2 \]

is in agreement with Ref. 25. At higher values of the energy, the rate of increase of the cross section is retarded and in the 550–1000 Mev region can be written as

\[ \sigma_n = (3.0 \pm 0.2) \times 10^{-4} \times 10^{27} \text{ cm}^2. \]

From a comparison of the data of Refs. 30 and 39, it follows that in the energy region 1000 Mev the cross section $\sigma_n$ goes through a broad maximum and thereafter slowly decreases with increasing proton energy.

In conclusion, we take the opportunity to express profound thanks to M. S. Kozodub, B. M. Ponecorvo, and L. I. Lapidus for the discussion of our results. We are indebted to M. M. Kaluzhkin for his advice in the construction of the apparatus.

---

15. B. Rosenfeld, Hubball, and Williams, Rev. Mod. Phys. 23, 409 (1951).
which can be as small as one likes for any $g_\alpha^2$ provided that $(L_q - L_j)/\kappa$ is large enough. Thus, when $(L_q - L_j)/\kappa \to \infty$ the series in Eq. (1) converges as quickly as one may like, and its zero term is in fact an exact solution.

From the standpoint of an expansion in a series like Eq. (1), the amplitude of meson-meson scattering is a quantity of the order of $g_\alpha^2$ (or $g_\alpha^2$ when there are two limits). Actually, the corresponding quantity for the simplest graphs in Fig. 1 has a value proportional to the product of $g_\alpha^2$ and a logarithmically divergent integral of the type $\int d^4 p / (2\pi)^4$, i.e., a value on the order of $g_\alpha^2$ (since $g_\alpha^2 \lesssim 1$).

FIG. 1

Accurate computation, with inclusion of only the most important logarithmic part of the resultant integral, shows that when the momenta are large, the sum of the contributions from the graphs in Fig. 1 depends only on the largest momentum, $k$, of the four momenta $k_1$, $k_2$, $k_3$, $k_4$ of the mesons, i.e.,

$$k = \max (|k_1|, |k_2|, |k_3|, |k_4|)$$

and can be written as

$$X(k_1, k_2, k_3, k_4) \approx (g_\alpha^2 / \alpha \ln k)^R_0 \left[ g_{\alpha}(L - 1) \right] \frac{1}{2}$$

where $R_0 = 24 (1 - Q^{2/3})$, $Q = 1 + (5g_\alpha^2 / 4\pi)(L - 1)$, and analogously for the symmetrical pseudoscalar theory

$$R_0 = \mu k, \quad \mu_\alpha = \frac{10}{3} \left( Q^{1/3} - 1 \right),$$

being dependent on indices (or variables) $\xi$ of the isotropic spin (1, 2, 3, 4) of the four mesons ($\xi_1 = 1, 2, 3, 4$). It is noteworthy that the amplitudes of the meson-meson theory in the sense of the logarithmically divergent part is eliminated when the contributions from the graphs in Fig. 1 are summed, so that they have a value of the order of $g_\alpha^2$ rather than of the order $g_\alpha^2$ as for the $g_\alpha(L - 1)$ type.

All of the more complicated graphs for meson-meson scattering can be categorized as "irreducible" graphs (in the sense of Fig. 1), if "irreducible" is taken to mean those graphs which consist only of nuclear squares joined by meson lines and subject to successive simplification to one of the graphs in Fig. 1. Such simplification consists of substituting, in some sequence, one square for two joined by two meson lines in the graphs in Fig. 1. For example, the graphs in Fig. 2 are "reducible," while those in Fig. 3 are "irreducible.

FIG. 2

FIG. 3

It is easily seen that such substitutions do not entail any alteration in the order of magnitude of a graph (symbolically, in the sense of a location in an expansion of type (1)), because in this case the number of points on the graph is reduced by four, the number of divergent integrals by two, and the contribution of the graph changes by a factor of the form $g_{\alpha}(L - 1)^2 = (g_{\alpha}(L - 1))^2$ which is of the order of unity. Therefore, any "irreducible" graph makes a contribution of the same order

$$g_{\alpha}(L - 1)R_0$$

$R_0$ is a dimensionless function and $n$ is the number of the "irreducible" graphs as contributions (3), (5a), and (5b). The contribution of the simplest graphs in Fig. 1. Analogously, simple calculation shows that "reducible" graphs have a value of a higher order in $k$.

*This was noted by Landau and does not apply to the mesonic theory, in which, in general, the elimination of the divergences in the graphs in Fig. 1. "Reducible" graphs even with two squares (Fig. 3) have a value on the order of $g_\alpha^2$ in comparison with the contributions from the graphs in Fig. 1.

\[ g_\alpha \] Therefore, if the scattering amplitude is written as

$$g_{\alpha}^2 / (2\pi)^2 P(k_1, k_2, k_3, k_4)$$

then $P$ will be represented, when the momenta are large, by a series of type (1)

$$P = P_0 (g_{\alpha}^2 / 4\pi)^2 + (g_{\alpha}^2 / 4\pi)^3 (g_{\alpha}^2 / 4\pi)^4 + \cdots$$

where the first term $P_0$ of this expansion (which, as was mentioned above, gives in fact the exact value of the scattering amplitude in meson theory) is determined by the infinite sum of the contributions from all the "reducible" graphs

\[ \sum (g_{\alpha}^2 / 4\pi)^n P_0 (k_1, k_2, k_3, k_4) \]

Below we consider the problem of computing this sum (the so-called "parquet" problem) where it is shown that the total sum is determined from a quantity $R_0$, with the aid of an integral equation, which in the case of large momenta is simple in form and can be solved exactly.

When $L$ is fixed, the magnitude of $P_0$ proves to be of the same order as that of $R_0$, despite the fact that the absolute value of the indefinitely alternating series, Eq. (7), increases rapidly as $n$ increases, whereby $P_0$ possesses the usual renormalization property, and in the limit as $L \to \infty$, can be renormalized without introducing intersection terms of the type $\lambda \varrho$ in the Hamiltonian.

The "parquet" problem is important for the evaluation of terms left out of the zeroth approximation in $g_{\alpha}^2$; this, as is well known, is an essential part of Pomeranchuk's work. In the theory advanced by Akhiezer, Galanin, and Khalatnikov, the equation for the vertex part operator (interacting operator) did not include (besides graphs with intersecting meson lines) graphs of the type shown in Fig. 4a and all the more complicated ones obtained from Fig. 4a by substituting
any graph reducible to a square for a nucleus square. When \( g^2 < 1 \) or \( g^2 < 1 \) in the theory with two limits, the contribution from the graphs in Fig. 4a is small (it contains an extra factor \((g^2/4n) R_0 \) of the order of \( g^2 \) in comparison with the contribution by the graph in Fig. 4b which was calculated by Abrikosov, et al.). However, the evaluation of the total contribution by all the more complicated graphs of the type shown in Fig. 4a, but with the square replaced by a complex graph of meson-meson scattering, depends essentially on the value of the sum

\[
P = \sum g^2 \delta_R
\]

if (this sum should turn out to be divergent, it is not permitted to neglect all the graphs of the type shown in Fig. 4a).

To avoid the difficulty of computing the sum

\[
P = \sum g^2 \delta_R
\]

in order to evaluate the contribution by all graphs of the type shown in Fig. 4a, Pomeranchuk analyzed a special type of limit process for point interaction, in which \( L_P/(L_P - L_A) \to 0 \) as \( L_A \to \infty \). In this case, all the complex graphs of meson-meson scattering (including the "reducible" ones) yield in the limit a contribution infinitely small in comparison with the simplest graphs shown in Fig. 1. For example, the contribution from graphs of the type a, b, c shown in Fig. 5, contains, in comparison with Eq. (3), twice the factor \( g^2 < 0 \) or \( L_A \to \infty \) and two divergent integrals over the nucleus and meson momenta that give the factor \( L_P L_A \), so that the total is a factor of the order of

\[
L_P L_A (g^2)^2 \sim L_P L_A (L_P - L_A)^2 \sim L_A / L_P \to 0,
\]

which goes to zero when \( L_A \to \infty \). In the limit \( P \to \infty \), i.e., the sum of the contributions from all of the complex graphs of the type shown in Fig. 5a coincides with the contribution from a single simple graph of the type shown in Fig. 4a.

The conclusion obtained below concerning the finiteness of \( P \) in agreement with Pomeranchuk's result which shows that the prediction that renormalized charge becomes zero in the pseudoscalar theory holds true only for a special type of approach to the limit, i.e., where \( L_P/(L_P - L_A) \to 0 \), but also for a more general case (if the sum \( P \) is a quantity of the same order as \( R_0 \), then the contribution from all graphs of the type shown in Fig. 4a is a quantity of the order of \( g^2 \) in comparison with the contribution from Fig. 4b and it may be disregarded, if \( g^2 < 1 \), i.e., if \( (L_P - L_A) \) is sufficiently small as \( L_A \to \infty \).

2. THE EXACT INTEGRAL EQUATION

We shall show that the sum (7) of the contribution by all of the "reducible" graphs* satisfies an exact integral equation whose form depends only on the value \( R_i(k_i, k_2, k_3, k_4) \) of the contribution from the simplest graphs in Fig. 1.

First we shall introduce the concept of reducible and irreducible graphs and derive a simple general relation (analogous to the Bysson-Schwinger equations for propagation functions or to the Bethe-Salpeter equation), which is satisfied by the total contribution \( P(k_1, k_2, k_3, k_4) \) of Eq. (6), of all general "reducible" and "irreducible" graphs.

Let us examine an arbitrary meson-meson scattering graph and call it reducible as regards the separation of meson lines ("ends") \( k_1 \) and \( k_2 \) from \( k_3 \) and \( k_4 \). If it can be divided, at least in one certain manner, into two parts connected with one another by only two meson lines, with the division made in such a way that lines \( k_1 \) and \( k_2 \) approach one part and \( k_3 \) and \( k_4 \) the other (e.g., the graphs in Figs. 5a, 5b, c), we shall call such graphs that do not possess this property reducible relative to "separation" of \( k_1 \) and \( k_2 \) from \( k_3 \) and \( k_4 \) (e.g., the graphs in Fig. 1 and Fig. 5b, c).

One and the same graph, depending on how the meson lines approach it, can be reducible or irreducible relative to the separation of \( k_1 \) and \( k_2 \) from \( k_3 \) and \( k_4 \) (the graph in Fig. 5a is reducible, while 5b and 5c are irreducible).

Let us designate by \( R'(k_1, k_2, k_3, k_4) \) the sum of the contributions from all of the graphs generally reducible in the indicated sense, and by \( F'(k_1, k_2, k_3, k_4) \) the sum of the contributions from all of the reducible ones. Since any graph is either reducible or irreducible, it is obvious that

\[
P = (9 / 2 \pi) \sum p = R'(k_1, k_2, k_3, k_4) + F'(k_1, k_2, k_3, k_4).
\]

The quantity \( P' \) is symmetrical for any transposition of the meson lines, and the values of \( R' \) and \( F' \) in Eq. (8) are unchanged if \( k_1 \) and \( k_2 \) or \( k_3 \) and \( k_4 \) are transposed or if \( k_1 \) and \( k_2 \) are interchanged with \( k_3 \) and \( k_4 \) (because they include a contribution from all of the graphs). For example, \( R' \) includes a contribution from both the graph in Fig. 5b, and that in Fig. 5c (the latter differs from 5b in the transposition of \( k_3 \) and \( k_4 \)), so that one may write

\[
R'(k_1, k_2, k_3, k_4) = R_i(k_1, k_2, k_3, k_4) + R_i(k_1, k_2, k_4, k_3),
\]

where the quantity \( R_i(k_1, k_2, k_3, k_4) \) includes the irreducible graphs only for arrangement of lines \( k_1, k_2, k_3, k_4 \) (thus, \( R' \) includes all graphs in Fig. 5b, while \( R_i(k_1, k_2, k_3, k_4) \) includes only one of them, e.g., Fig. 5b).

In this case we are not difficult to see that \( P' \) and \( R' \) are connected by the following integral equation

\[
P'(k_1, k_2, k_3, k_4) = R'(k_1, k_2, k_3, k_4) + F'(k_1, k_2, k_3, k_4).
\]

Since any particular graph may be adjacent to \( k_1 \) and \( k_2 \), by the same token

\[
\sum_{km} p_{km} (l-l', l'-l) = R'(k_1, k_2, l, l'),
\]

since only irreducible graphs are, by definition, adjacent to \( k_1 \) and \( k_2 \). Furthermore, it is evident that

\[
\sum_{km} p_{km} (k_1, k_2, k_3, k_4) = 2F'(k_1, k_2, k_3, k_4).
\]

The factor 2 appears because half of all the graphs from the sum in Eq. (13) that enter into \( R_i(k_1, k_2, k_3, k_4) \) will, on account of the symmetry of \( P' \) and \( F' \), give exactly the same set of all reducible graphs as will the other half of the sum in Eq. (12), which enters into \( R_i(k_1, k_2, k_3, k_4) \).

By utilizing equalities (10) to (14) we obtain, after summing both halves members of (11) over \( k \) and \( m \).

*Since \( k \) or \( k \) always represent the momentum entering into any graph, then \( k_1 \) or \( k_2 \) or \( k_3 \) or \( k_4 \) must be zero, while the momenta \( l \) and \( l' \) enter into the part adjacent to \( k_1 \) and \( k_2 \), come out of the other part and are written in \( R'_i \) with a minus sign.
necessarily reducible as regards the separation of any one pair of meson lines from another, e_i \hspace{1em} e_j \hspace{1em} e_k \hspace{1em} e_l \hspace{1em} \ldots

\text{It is obvious that Eq. (15) together with Eq. (9) gives Eq. (10).}

Now let us consider only the "reducible" graphs and designate by R, F and \bar{F} the corresponding sums determined by them only. With respect to these graphs, all of the considerations that led to Eq. (8), (15) and (10) can be repeated literally; in an analogous fashion we obtain

\begin{equation}
P(h_1, h_2, h_3, h_4) = R(h_1, h_2, h_3, h_4),
\end{equation}

\begin{equation}
F(h_1, h_2, h_3, h_4) = \frac{-g^2}{2m^2} \int \left[ R(h_1, h_2, h_1', h_1') \right] d^4 \lambda.
\end{equation}

These relations permit one to solve the problem stated above, if one bears in mind that the "reducible" graph, if it is not the simplest one (Fig. 1), is reducible in one of the three graphs or functions, i.e., because it is reducible as regards the separation of any one pair of lines from another, it is then irreducible to any other division of meson lines into pairs. This is made clear by any analysis of the graphs, as in Fig. 5 to which any complex reducible graph may be "reduced". Therefore, by including the independent contribution \bar{R}_0 from the simplest graphs in Fig. 1, we obtain

\begin{equation}
P(h_1, h_2, h_3, h_4) = R(h_1, h_2, h_3, h_4) + F(h_1, h_2, h_3, h_4) + \bar{R}_0(h_1, h_2, h_3, h_4).
\end{equation}

(17)

(18)

(19)

\text{The above equations are then}

\begin{equation}
P(h_1, h_2, h_3, h_4) = R(h_1, h_2, h_3, h_4) + F(h_1, h_2, h_3, h_4) + \bar{R}_0(h_1, h_2, h_3, h_4).
\end{equation}

(17)

If we put in Eq. (20) \eta = \zeta and \zeta = \xi = \xi, we obtain two equations whose simultaneous solution determines the functions \Phi(\xi) and \phi(\xi, \eta, \zeta) = \Phi(\xi).

\begin{equation}
P(h_1, h_2, h_3, h_4) = R(h_1, h_2, h_3, h_4) + F(h_1, h_2, h_3, h_4) + \bar{R}_0(h_1, h_2, h_3, h_4).
\end{equation}

(17)

(18)

\text{and}

\begin{equation}
P(h_1, h_2, h_3, h_4) = R(h_1, h_2, h_3, h_4) + F(h_1, h_2, h_3, h_4) + \bar{R}_0(h_1, h_2, h_3, h_4).
\end{equation}

(17)

(18)

\text{where}

\begin{equation}
P(h_1, h_2, h_3, h_4) = R(h_1, h_2, h_3, h_4) + F(h_1, h_2, h_3, h_4) + \bar{R}_0(h_1, h_2, h_3, h_4).
\end{equation}

(17)

(18)

\text{In the case of the symmetrical theory the equations look even more awkward. In this case the isotopic meson spin variables, \alpha, \beta, can be left out of the calculations if one desires asymptotic solutions for Eq. (19) of the form}

\begin{equation}
F(h_1, h_2, h_3, h_4) = \Phi(\xi, \eta, \zeta) + F_0(h_1, h_2, h_3, h_4)
\end{equation}

\text{where the function \Phi(\xi, \eta, \zeta) is then the same as in Eq. (20).}

\begin{equation}
F(h_1, h_2, h_3, h_4) = \Phi(\xi, \eta, \zeta) + F_0(h_1, h_2, h_3, h_4).
\end{equation}

(17)

(18)


The function \( P(x, y) \) is given by

\[
P(x, y) = \frac{1}{2} \left[ \left( x^2 + y^2 \right) - \frac{1}{2} \right] - \frac{1}{3} \left[ \left( x^2 + y^2 \right) - \frac{1}{2} \right]^2 + \Phi(x, y)
\]

where \( \Phi(x, y) = \frac{1}{2} \left\{ \begin{array}{c}
\left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} \\
\left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}
\end{array} \right\} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

To obtain the second equation in Eq. (24), the function \( P(x, y) \) and \( \Phi(x, y) \) can be substituted into the formula for the second equation.

\( \Phi(x, y) \) is given by

\[
\Phi(x, y) = \frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then substituted into the formula for the second equation in Eq. (24), resulting in

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\Phi(x, y) = \frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]

The function \( \Phi(x, y) \) is then simplified to

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2},
\]

and

\[
\frac{1}{2} \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2} + \left( x^2 + y^2 \right) \left( \frac{1}{2} \right) - \frac{1}{2}.
\]
where the quantity

\[ a_{n_1 n_2 ... n_{\ell-1}}(l) = \left( -\frac{g^2}{8\pi^2} \right)^{\ell-1} \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \right) \]

is determined in a manner exactly analogous with Eq. (29), i.e., it may be regarded as the contribution from the part of the reducible graph under consideration that adjoins lines \( k_i \), \( l_i \) and \( l_i' \) (part I in Fig. 7), which would be made if the momenta \( k_i \) and \( l_i' \) were of the same order as \( l_i \). Analogously, \( a_{p, q, r, s, t} \) may be regarded as the contribution that would come from part II in Fig. 7 if \( k_i \) and \( l_i' \) were of the same order as \( l_i \).

It is not difficult to see that, in analogy with Eq. (12)

\[ \sum_{n_1, n_2, ... n_{\ell-1}} 2^{\ell-1} \sum_{n_j = 0} \frac{a_{n_1, n_2, ... n_{\ell-1}}(l)}{a(l)} = P(l), \]

where \( P(l) \) is the desired value for the sum of all the reducible graphs provided that all the momenta are of the same order. In exactly the same way as in Eq. (14) we obtain

\[ \sum_{n_1, n_2, ... n_{\ell-1}} 2^{\ell-1} \sum_{n_j = 0} \frac{a_{n_1, n_2, ... n_{\ell-1}}(l)}{a(l)} = P(l). \]

The negative powers of 2 in the left side of these equations are due to the fact that when we sum over all possible irreducible parts, i.e., over \( n_j (j = 0, ... , N) \), we obtain (in the last case) as identical resulting graphs.

Using these equalities, we obtain, by the summation of Eq. (29) over all types (or numbers) of \( n_j \) graphs in each irreducible part and over the number \( N \) of these parts,

\[ F(l) = \left( -\frac{g^2}{8\pi} \right)^{\ell} \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \right) \]

(32)

For the neutral pseudoscalar theory we obtain according to Eq. (32) a simple integral equation,

\[ P(l) = R(l) - \frac{g^2}{8\pi^2} \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \]

(33a)

where \( R(l) \) is determined in Eq. (33a).

Introducing (as in Eq. (27b)) a more convenient variable

\[ x = \left[ 1 + \left( \frac{5g^2}{4\pi} \right) (L - 1) \right]^{-\alpha} = Q^{-\alpha}, \]

we can rewrite Eq. (33a) as

\[ P(x) = 24 \left( 1 - x \right) - \frac{3}{2} \frac{1}{x^2} \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \]

(34a)

\[ x \sum_{\ell = 1}^{\infty} \frac{1}{\ell!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) = P(\ell) \]

(35a)

\[ \sum_{\ell = 1}^{\infty} \frac{1}{\ell!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) = P(\ell) \]

(36a)

In the case of the symmetrical theory Eq. (32) should be substituted in Eq. (33) and summed over the indices of the isospin; this will give

\[ F(0, 1, 2, 3) \approx [2b_1 - 2b_2] \approx [2b_1 - 2b_2] \]

\[ \times \left( -\frac{g^2}{8\pi} \right)^{1/2} \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \]

where (see footnote* following Eq. (20)) \( d(l) = Q^{-\alpha} \). In conformity with Eq. (17), we then obtain (upon dividing through \( \delta_1 \))

\[ P(x) = \left( x - \frac{1}{2} \right) \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \]

(33b)

\[ P(x) = \left( x - \frac{1}{2} \right) \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \]

(34b)

or, if we introduce the variable \( x \), make use of Eq. (33b), and integrate, we obtain

\[ P(x) = \left( \frac{1}{l!} - \frac{1}{l!} x^{-\alpha} \right) \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \]

(35b)

Thus, the total sum \( P(x) \) of the reducible graphs is a finite quantity of the same order as the contribution \( \pi \) from the simplest graphs in Fig. 1 (where \( \pi \approx \pi \), Eq. (55a) for example, differs from Eq. (29) only in the factor \( 3/11 \)).

Eq. (33a) and (33b) could, it would seem, also be renormalized mathematically directly from Eq. (29) to (36). However, we were unable to do this. The value of Eq. (35a) for \( x = 1 \) and \( x = 1 \) coincide with the value of Eq. (29) derived directly from Eq. (26).

4. THE RENORMALIZATION PROPERTIES OF THE AMPLITUDE FOR MESON-MESON SCATTERING

In the conventional scheme for charge renormalization the amplitude for meson-meson scattering

\[ \left( g_i^2 / 4 \pi \right) P (k_i, k_j, k_k, k_l). \]

which corresponds to graphs with four external meson lines, is multiplied by a factor \( Z_i^2 \) (where \( D = Z_i D_1 \), \( Z_i \) for each external meson line. Thus every renormalized expression will contain the quantity

\[ Z_i^2 \left( g_i^2 / 4 \pi \right) P (k_i, k_j, k_k, k_l). \]

Since, according to Ref. 2, \( g_i^2 / 4 \pi \) is determined, \( Z_i \) (D = \( \theta \)) for the neutral theory and \( Z_i = \theta^\alpha \) for the symmetrical theory, we obtain in both cases

\[ Z_i^2 \left( g_i^2 / 4 \pi \right) P (k_i, k_j, k_k, k_l). \]

where, according to Eq. (27b), \( x = \theta^{-\alpha} \) for the neutral theory and \( Q^{-\alpha} \) for the symmetrical one. Thus, after renormalization of the charge, instead of \( P(x) \) all the equations of the form

\[ P(x) = P(x) \]

(33c)

or, if we introduce the variable \( x \), make use of Eq. (33b), and integrate, we obtain

\[ P(x) = \left( x - \frac{1}{2} \right) \sum \frac{1}{l!} \left( \frac{1}{l} \right) d_1 d_2 ... d_{\ell-1} d_{\ell-1} a_{n_1, n_2, ... n_{\ell-1}}(l) \]

(34b)

If we set \( x = x_0 \) where \( x_0 = \theta^{-\alpha} \) for the neutral theory and \( x_0 = Q^{-\alpha} \) for the symmetrical theory, and where

\[ \theta = 1 + \sqrt{g_i^2 / 4\pi} \]

we see that \( \theta = \theta \) for \( \theta = 0 \) and \( \theta = 0 \) for \( \theta = \theta \).

If \( P(x) \) is given by Eq. (35a) or (35b) with \( Q = \theta \) (i.e., \( P(x) \) for \( \theta = \theta \) for Eq. (35a) and \( P(x) \) for \( \theta = \theta \) for Eq. (35b)),

\[ P(x) = P \]

(38)

Thus by harmonizing the right hand side of the same equation as Eq. (37), and where \( \lambda_0 \) and \( \lambda_0 \) are connected by the equality

\[ \frac{1 + x_0^2}{1 - \lambda_0 x_0} \]

(39)
Nonlinearity of the Field in Conformal Reciprocity Theory

A. Popovici

Bucharest, Romania

(Submitted to JETP editor February 16, 1956)


The first version of Born and Infeld's nonlinear electrodynamics and the variational of the gravitational constant are deduced from the conformally covariant gravitational equations derived from a certain generalized reciprocity law which is based on group theory and yields a nonlocal field theory. A correspondence principle is established between relativity theory and reciprocity theory.

1. FIELD EQUATIONS

Let $\mathbf{E}$ be Einstein's gravitational constant, $\kappa = \frac{\kappa}{\kappa}$ and $\kappa = \frac{\kappa}{\kappa}$ be the element of interval with the metric $\kappa = \frac{\kappa}{\kappa}$, $\kappa = \frac{\kappa}{\kappa}$, and of weight zero with respect to the $\kappa = \frac{\kappa}{\kappa}$, $\kappa = \frac{\kappa}{\kappa}$, and $\kappa = \frac{\kappa}{\kappa}$, which is obtained directly from the conformal covariant gravitational equations derived from a certain generalized reciprocity law which is based on group theory and yields a nonlocal field theory. A correspondence principle is established between relativity theory and reciprocity theory.

The $\Gamma_{ik}$, $\Gamma_{ik}$, and $\Gamma_{ik}$, (as well as the $\kappa_{ik}$) depend only on the ratio of the $\kappa_{ik}$ and therefore on those of the $\kappa_{ik}$, $\kappa_{ik}$, and $\kappa_{ik}$, $\kappa_{ik}$, which are obtained directly from the conformal covariant gravitational equations derived from a certain generalized reciprocity law which is based on group theory and yields a nonlocal field theory. A correspondence principle is established between relativity theory and reciprocity theory.