**Meaning of Matrix Multiplication**

1. In this problem we will show that multiplication by the matrix

\[
A = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

acts by rotating vectors 45° counterclockwise. As usual, we write the vector \( \mathbf{v} = xi + yj \) as a column vector \( \begin{pmatrix} x \\ y \end{pmatrix} \).

a) Show that the length of \( Av \) is the same as the length of \( v \).

b) Use the dot product to show the angle between \( v \) and \( Av \) is \( \pi/4 \) radians.

c) Use the cross product to show \( Av \) is \( \pi/4 \) radians counterclockwise from \( v \).

**Answer:**

a) \[
Av = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x-y}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \end{pmatrix}.
\]

This has length \( \sqrt{\frac{(x-y)^2}{2} + \frac{(x+y)^2}{2}} = \sqrt{x^2 + y^2} \). That is, we have shown \( |Av| = |v| \) as required.

b) Using the expression for \( Av \) found in part (a) we compute the dot product

\[
Av \cdot v = \left\langle \frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right\rangle \cdot (x, y) = \frac{(x^2 + y^2)}{\sqrt{2}}.
\]

By part (a) we know \( |Av| = |v| = \sqrt{x^2 + y^2} \). So the cosine of the angle between the two vectors is

\[
\frac{Av \cdot v}{|Av||v|} = \frac{1}{\sqrt{2}} = \cos(\pi/4).
\]

c) We compute the cross product

\[
\mathbf{v} \times Av = \begin{vmatrix}
i & j & k \\
x & y & 0 \\
(x-y)/\sqrt{2} & (x+y)/\sqrt{2} & 0
\end{vmatrix} = \frac{x^2 + y^2}{\sqrt{2}} \mathbf{k}.
\]

Since the coefficient of \( \mathbf{k} \) is positive the right hand rule tells us \( Av \) is counterclockwise from \( v \).