Components and Projection

If \( \mathbf{A} \) is any vector and \( \mathbf{\hat{u}} \) is a unit vector then the *component* of \( \mathbf{A} \) in the direction of \( \mathbf{\hat{u}} \) is

\[
\mathbf{A} \cdot \mathbf{\hat{u}}.
\]

(Note: the component is a scalar.)

If \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{\hat{u}} \) then since \( |\mathbf{\hat{u}}| = 1 \)

\[
\mathbf{A} \cdot \mathbf{\hat{u}} = |\mathbf{A}||\mathbf{\hat{u}}|\cos\theta = |\mathbf{A}|\cos\theta.
\]

The figure shows that geometrically this is the length of the leg of the right triangle with hypotenuse \( \mathbf{A} \) and one leg parallel to \( \mathbf{\hat{u}} \).

We also call the leg parallel to \( \mathbf{\hat{u}} \) the *orthogonal projection* of \( \mathbf{A} \) on \( \mathbf{\hat{u}} \).

For a non-unit vector: the component of \( \mathbf{A} \) in the direction of \( \mathbf{B} \) is simply the component of \( \mathbf{A} \) in the direction of \( \mathbf{\hat{u}} = \frac{\mathbf{B}}{|\mathbf{B}|} \). (\( \mathbf{\hat{u}} \) is the unit vector in the same direction as \( \mathbf{B} \).)

**Example:** Find the component of \( \mathbf{A} \) in the direction of \( \mathbf{B} \).

i) \( |\mathbf{A}| = 2, \ |\mathbf{B}| = 5, \ \theta = \pi/4 \).

**Answer:** Referring to the figure above: the component is \( |\mathbf{A}|\cos\theta = 2\cos(\pi/4) = \sqrt{2} \).

Note, the length of \( \mathbf{B} \) given is irrelevant, since we only care about the unit vector parallel to \( \mathbf{B} \).

ii) \( \mathbf{A} = \mathbf{i} + 2\mathbf{j}, \ \mathbf{B} = 3\mathbf{i} + 4\mathbf{j} \).

**Answer:** Unit vector in direction of \( \mathbf{B} \) is \( \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \) \( \Rightarrow \) component is \( \mathbf{A} \cdot \mathbf{B}/|\mathbf{B}| = 3/5 + 8/5 = 11/5 \).

iii) Find the component of \( \mathbf{A} = (2, 2) \) in the direction of \( \mathbf{\hat{u}} = (-1, 0) \)

**Answer:** The vector \( \mathbf{\hat{u}} \) is a unit vector, so the component is \( \mathbf{A} \cdot \mathbf{\hat{u}} = (2, 2) \cdot (-1, 0) = -2 \). The negative component is okay, it says the projection of \( \mathbf{A} \) and \( \mathbf{\hat{u}} \) point in opposite directions.

We emphasize one more time that the component of a vector is a *scalar*. 

\[
\begin{align*}
\text{A} & \qquad \theta \\
\mathbf{\hat{u}} & \\
\end{align*}
\]