Problem Set 1

June 12, 2005

1. Show that if there are \( n \) people at a party, then two of them know the same number of people.

2. (a) Find all \( n \) such that 7 is a factor of \( 2^n - 1 \).
   (b) Prove there is no \( n \) such that 7 is a factor of \( 2^n + 1 \).

3. Determine all non-negative integral solutions \((n_1, n_2, \ldots, n_{14})\), apart from permutations, of the Diophantine equation

\[
 n_1^4 + n_2^4 + \ldots + n_{14}^4 = 159,999.
\]

4. Circles \( \omega_1 \) and \( \omega_2 \) are externally tangent at \( T \). Line \( \ell \) is tangent to \( \omega_2 \) at \( X \) and meets \( \omega_1 \) at \( A \) and \( B \). Line \( XT \) meets \( \omega_1 \) again at \( S \). Point \( C \) lies on arc \( ST \) of \( \omega_1 \) and \( CM \) is tangent to \( \omega_2 \) at \( M \). Lines \( SC \) and \( XM \) intersect at \( I \).
   (a) Prove that quadrilateral \( CTIM \) is cyclic.
   (b) Prove that \( I \) is the excenter opposite \( A \) of triangle \( ABC \).

5. Let \( P \) be a point on the arc \( AB \) of the circumcircle of \( ABC \) that does not contain \( C \). Let \( I_1 \) be the incenter of \( PAC \) and \( I_2 \) be the incenter of \( PBC \). Show that the circumcircle of \( PI_1I_2 \) passes through a fixed point of the circumcircle of \( ABC \).

6. Let \( f : \mathbb{C} \to \mathbb{C} \) be a polynomial defined by \( f(z) = z^n + a_1z^{n-1} + \ldots + a_n \) with roots \( z_1, z_2, \ldots, z_n \) such that \( \sum_{k=1}^{n} |a_k|^2 \leq 1 \). Prove that \( \sum_{k=1}^{n} |z_k|^2 \leq n \).

7. Let \( u_1, u_2, \ldots, u_n \) be \( n \) 2-dimensional vectors such that none of their lengths is greater than 1 and \( u_1 + u_2 + \ldots + u_n = 0 \). Show that you can find a permutation \( v_1, v_2, \ldots, v_n \) of \( u_1, u_2, \ldots, u_n \) such that none of the sums \( v_1 + v_2, \ldots, v_1 + v_2 + \ldots + v_n \) have lengths greater than \( \sqrt{n} \).

8. Let \( O \) be the center of a circle circumscribed about convex \( 2n \)-gon \( A_1A_2 \cdots A_{2n} \). Prove

\[
 \left| \sum_{i=1}^{n} A_{2i-1}A_{2i} \right| \leq 2 \sin \frac{\angle A_1OA_2 + \angle A_3OA_4 + \cdots + \angle A_{2n-1}OA_{2n}}{2}.
\]