Inventory Strategies for International Non-Profit Healthcare Organizations

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The challenge...

- International Healthcare NGO operating large hospitals in Haiti
- Limited human/IT resources
  => Annual order placement
- Limited storage space
  => Difficult to manage a year’s worth of inventory
- Possible solution: Blanket orders
Blanket Orders (Standing Orders)

- Long-term commitment
  - Order large quantity in the beginning of the year
- Discounted rate
- Frequent fulfillment
  - Predetermined delivery dates
  - Delivery amount can be different for each period

What is the optimal blanket order delivery amount?
## Related Work: Stochastic Demand

<table>
<thead>
<tr>
<th>Model</th>
<th>Newsvendor</th>
<th>Scarf (1960)</th>
<th>Our work</th>
</tr>
</thead>
<tbody>
<tr>
<td># Horizon</td>
<td>1</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>Setting</td>
<td>Stochastic demand</td>
<td>Stochastic i.i.d.</td>
<td>Stochastic indept.</td>
</tr>
<tr>
<td></td>
<td>Stochastic backlogging</td>
<td>Backlogging</td>
<td>Backlogging</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Total order = Q</td>
<td></td>
</tr>
<tr>
<td>Costs</td>
<td>Overstock cost h</td>
<td>Holding cost h</td>
<td>Holding cost h</td>
</tr>
<tr>
<td></td>
<td>Understock cost p</td>
<td>Shortage cost p</td>
<td>Shortage cost p</td>
</tr>
<tr>
<td></td>
<td>Order set-up cost K</td>
<td></td>
<td>No set-up cost</td>
</tr>
<tr>
<td>Solution</td>
<td>[ q^* = F^{-1} \left( \frac{p}{h+p} \right) ]</td>
<td>(s,S) policy</td>
<td>[ \sum_{n=1}^{N} q_n^* = F_n^{-1} \left( \frac{p}{h+p} \right) ]</td>
</tr>
<tr>
<td>Method</td>
<td>Optimality conditions</td>
<td>DP</td>
<td>Optimality conditions</td>
</tr>
<tr>
<td></td>
<td>K-convexity</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Blanket Order: Model

- **Given:** $X_n$, total demand in first $n$ periods, $\sim f_n$
  - $Q$, total order quantity (by contract)
  - $h$, per unit holding cost
  - $p$, per unit shortage cost

- **Choose:** $q_n$, order quantity in each period, $n = 1, \ldots, N$

\[
\begin{align*}
\min_q & \quad \sum_{n=1}^{N} c_n(q) \\
\text{s.t.} & \quad \sum_{n=1}^{N} q_n = Q \\
& \quad q_n \geq 0 \quad \forall n
\end{align*}
\]

- total expected cost
- total order quantity $= Q$
- order quantities nonnegative
Cost function

- Let
  \[ Q_n = \sum_{i=1}^{n} q_i \quad \Rightarrow \quad \frac{d}{dq_i} Q_n = \begin{cases} 
  1, & n \geq i \\
  0, & n < i 
\end{cases} \]

- Expected cost incurred at period \( n \):
  \[
  c_n(q) = E[h(Q_n - X_n)_+ + p(X_n - Q_n)_+] \\
  = h \int_{0}^{Q_n} (Q_n - x) f_n(x) \, dx + p \int_{Q_n}^{\infty} (x - Q_n) f_n(x) \, dx
  \]
  \[\text{holding} \quad \text{shortage}\]

- Derivative of expected cost:
  \[
  \frac{d}{dq_i} c_n(q) = \begin{cases} 
  hF_n(Q_n) + p[1 - F_n(Q_n)], & n \geq i \\
  0, & n < i 
\end{cases}
  \]
Optimal Solution: Structure

\[ F_n(Q_n) \]

\[ \rho = \frac{p}{h+p} \]

\[ n_1 < n_2 < n_3 \]

\[ q_n = 0 \text{ for } n > 3 \]
Optimal Solution: Analysis

- **Lagrangian:**
  \[
  L(q, \lambda, \mu) = \sum_{n=1}^{N} c_n(q) + \lambda \left( \sum_{n=1}^{N} q_n - Q \right) - \sum_{n=1}^{N} \mu_n q_n
  \]

- **KKT optimality conditions:** \( \forall i \),
  - **Stationarity:**
    \[
    \frac{d}{dq_i} L(q, \lambda, \mu) = \sum_{n \geq i} \left( hF_n(Q_n) - p[1 - F_n(Q_n)] \right) + \lambda - \mu_i = 0 \quad \forall i
    \]
  - **Complementary slackness:**
    \[ q_i \mu_i = 0 \]
  - **Dual feasibility:**
    \[ \mu_i \geq 0 \]
**Optimal Solution: Analysis**

- **Key observation:**
  
  if $\mu_{\bar{n}} > 0$ for some $\bar{n}$, then $\mu_n > 0 \quad \forall n \geq \bar{n}$.

- **Proof:**
  
  \[
  \frac{dL}{dq_i} = \sum_{n \geq i} (hF_n(Q_n) - p[1 - F_n(Q_n)]) + \lambda - \mu_i = 0 \quad \forall i
  \]

  \[
  \Rightarrow \lambda = \sum_{n \geq i} ((h + p)F_n(Q_n) - p) - \mu_i \quad \forall i
  \]

  Suppose $\mu_{n-1} = 0, \mu_n > 0 \quad \Rightarrow q_n = 0 \Rightarrow Q_n = Q_{n-1}$

  \[
  \left\{ \begin{array}{l}
  (h + p)F_{n-1}(Q_{n-1}) - p = \mu_{n-1} - \mu_n < 0 \\
  \end{array} \right. \quad \downarrow
  \]

  \[
  \Rightarrow \begin{cases} 
  F_n(Q_n) = F_n(Q_{n-1}) < F_{n-1}(Q_{n-1}) \\
  (h + p)F_n(Q_n) - p = \mu_n - \mu_{n+1} > -\mu_{n+1}
  \end{cases}
  \]

  \[
  \Rightarrow \mu_{n+1} > 0
  \]
Optimal Solution: Analysis

- **Key observation:**
  \[
  \text{if } \mu_{\bar{n}} > 0 \text{ for some } \bar{n}, \text{ then } \mu_n > 0 \quad \forall n \geq \bar{n}.
  \]

- **Due to complementary slackness,**
  \[
  q_n = 0 \quad \forall n \geq \bar{n}
  \]

\[
\Rightarrow \quad Q_{\bar{n}-1} = \sum_{i=1}^{\bar{n}-1} q_i = Q
\]

\[
(h + p)F_n(Q_n) - p = \mu_n - \mu_{n+1} = 0 \quad \forall n < \bar{n} - 1
\]

\[
\Rightarrow \quad Q_n = F_n^{-1}(\rho) \quad \text{where } \rho = \frac{p}{h+p}
\]

\[
\Rightarrow \quad q_n = Q_n - Q_{n-1} = F_n^{-1}(\rho) - F_{n-1}^{-1}(\rho) \quad \forall n = 1, \ldots, \bar{n} - 2
\]

\[
q_{\bar{n}-1} = Q_{\bar{n}-1} - Q_{\bar{n}-2} = Q - F_{\bar{n}-2}^{-1}(\rho)
\]
Optimal Solution: Structure

\[ F_n(Q_n) \]

\[ \rho = \frac{p}{h+p} \]

\[ q_n = Q_n - Q_{n-1} = F_{n-1}^{-1}(\rho) - F_{n-2}^{-1}(\rho) \quad \forall n = 1, \ldots, \bar{n} - 2 \]

\[ q_{\bar{n}-1} = Q_{\bar{n}-1} - Q_{\bar{n}-2} = Q - F_{\bar{n}-2}^{-1}(\rho) \]

\[ q_{\bar{n}} = \cdots = q_N = 0 \]
Numerical Experiments

- Given: $X_n$, Gaussian demand forecasts (cv= $\frac{\sigma}{\mu}$=0.2)
  
  - $Q$, perturbation of expected total demand
  
  - $h = 1$
  
  - $p = 5$
  
  \[ \rho = \frac{p}{h+p} = \frac{5}{6} \]

- $N = 12$ periods

- 1000 samples of realized demand

- Compare:
  - Optimal blanket order
  - Proportional allocation (benchmark)
Numerical Example 1 (Q< Demand)

Cumulative (total demand = 6.24, Q = 4.56)

- Optimal (cost = 21.97 = 1.97 + 20.0)
- Benchmark (cost = 52.98 = 0.01 + 52.97)
Numerical Example 1 (Q<Demand)

Per period (total demand = 6.24, Q=4.56)

- Optimal
- Benchmark

Legend
Numerical Example 2 (Q>Demand)

Cumulative (total demand = 6.88, Q = 8.45)

- Optimal (cost = 6.82 = 5.18 + 1.64)
- Benchmark (cost = 10.58 = 10.47 + 0.12)
Numerical Example 2 (Q>Demand)

Per period (total demand = 6.88, Q = 8.45)

- Optimal
- Benchmark

Amount

Period
Conclusions & Future Work

- Blanket orders help solve inventory challenge!
  - Long-term commitment + frequent fulfillment
  - Simple optimal solution: Newsvendor-like policy till run out
  - Uses cumulative demand forecasts + KKT optimality conditions

- Next steps:
  - Test with real data; integrate with forecasting engine; ship
  - Applications in other domains?

- Extensions of model:
  - Sensitivity analysis
  - Advanced holding/shortage cost models
    - Different holding/shortage costs for each period
    - Holding cost per unit time and unit inventory (=> quadratic)
    - Nonlinear shortage cost with respect to unfulfilled time
  - Moving horizon implementation; allow forecast/quantity updates
  - Multiple locations

Thank you!

Questions? Comments? → anniecia@mit.edu
References

- H. Scarf, *The Optimalities of (s,S) Policies in the Dynamic Inventory Problem.*

- E. L. Porteus, *Stochastic Inventory Theory.*