Variance bias derivation (optional)

This derivation requires the identity \( \text{var}[x] = E[x^2] - E[x]^2 \), along with the fact that if \( x \) and \( y \) are independent, then \( E[xy] = E[x]E[y] \). For convenience, let \( E[x_i^2] = E[x^2] \), since it’s the same for all \( x_i \).

\[
E[\hat{\sigma}^2] = E \left[ \frac{1}{n-1} \sum_i (x_i - \bar{x})^2 \right]
\]
\[
= \frac{1}{n-1} \sum_i E[x_i^2 - 2x_i\bar{x} + \bar{x}^2]
\]
\[
= \frac{1}{n-1} \sum_i \left( E[x_i^2] - \frac{2}{n} E \left[ x_i \sum_j x_j \right] + \frac{1}{n^2} \mathbb{E} \left[ \left( \sum_j x_j \right)^2 \right] \right)
\]
\[
= \frac{1}{n-1} \sum_i \left( E[x_i^2] - \frac{2}{n} E \left[ x_i^2 \right] - \frac{2}{n} E \left[ x_i \sum_{j \neq i} E \left[ x_j \right] \right] + \frac{1}{n^2} \sum_j E \left[ x_j^2 \right] + \frac{1}{n^2} \mathbb{E} \left[ \sum_{j \neq i} x_j x_k \right] \right)
\]
\[
= \frac{n}{n-1} E[x^2] - \frac{2}{n} n E[x^2] - \frac{2}{n(n-1)} n(n-1) \mu^2 + \frac{1}{n^2} n(n-1) E[x^2] + \frac{1}{n^2} n(n-1) E[x]^2
\]
\[
= E[x^2] - E[x]^2
\]
\[
= \text{var}[x]
\]