Your name is: SOLUTIONS

Please circle your recitation:

1) M2 2-131 I. Ben-Yaacov 2-101 3-3299 pezz
2) M3 2-131 I. Ben-Yaacov 2-101 3-3299 pezz
3) M3 2-132 A. Oblomkov 2-092 3-6228 oblomkov
4) T11 2-132 A. Oblomkov 2-092 3-6228 oblomkov
5) T12 2-132 I. Pak 2-390 3-4390 pak
6) T1 2-131 B. Santoro 2-085 2-1192 bsantoro
7) T1 2-132 I. Pak 2-390 3-4390 pak
8) T2 2-132 B. Santoro 2-085 2-1192 bsantoro
9) T2 2-131 J. Santos 2-180 3-4350 jsantos
1 (40 pts.) This question deals with the following symmetric matrix $A$:

$$A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
1 & -1 & 0
\end{bmatrix}$$

One eigenvalue is $\lambda = 1$ with the line of eigenvectors $x = (c, c, 0)$.

(a) That line is the nullspace of what matrix constructed from $A$?

(b) Find (in any way) the other two eigenvalues of $A$ and two corresponding eigenvectors.

(c) The diagonalization $A = S\Lambda S^{-1}$ has a specially nice form because $A = A^T$. Write all entries in the three matrices in the nice symmetric diagonalization of $A$.

(d) Give a reason why $e^A$ is or is not a symmetric positive definite matrix.

Solution:

(a) The eigenvectors for $\lambda = 1$ make up the nullspace of $A - I$.

(b) First method: $A$ has trace 2 and determinant $-2$. So the two eigenvalues after $\lambda_1 = 1$ will add to 1 and multiply to $-2$. Those are $\lambda_2 = 2$ and $\lambda_3 = -1$.

Second method: Compute $\det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + \lambda - 2$ and find the roots 1, 2, $-1$:

(divide by $\lambda - 1$ to get $\lambda^2 - \lambda - 2 = 0$ for the roots $\lambda_2$ and $\lambda_3$).

Eigenvectors: $\lambda_2 = 2$ has $x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\lambda_3 = -1$ has $x_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

(c) Every symmetric matrix has the nice form $A = Q\Lambda Q^T$ with orthogonal matrix $Q$. The columns of $Q$ are orthonormal eigenvectors. (They could be multiplied by $-1$.)

$$Q = \begin{bmatrix}
1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\
1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\
0 & 1/\sqrt{3} & -2/\sqrt{6}
\end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{bmatrix}.$$

(d) $e^A$ is symmetric and all its eigenvalues $e^\lambda$ are positive—so $e^A$ is positive definite.
2 (30 pts.)

(a) Find the eigenvalues and eigenvectors (depending on $c$) of

$$ A = \begin{bmatrix} 0.3 & c \\ 0.7 & 1 - c \end{bmatrix}. $$

For which value of $c$ is the matrix $A$ not diagonalizable (so $A = SAS^{-1}$ is impossible)?

(b) What is the largest range of values of $c$ (real number) so that $A^n$ approaches a limiting matrix $A^\infty$ as $n \to \infty$?

(c) What is that limit of $A^n$ (still depending on $c$)? You could work from $A = SAS^{-1}$ to find $A^n$.

Solution:

(a) Both columns add to 1. As we know for Markov matrices, $\lambda = 1$ is an eigenvalue. From trace($A$) = $0.3 + (1 - c)$ the other eigenvalue is $\lambda = 0.3 - c$. Check: det $A = \lambda_1\lambda_2 = (1)(0.3 - c)$ is correct.

The eigenvector for $\lambda = 1$ is in the nullspaces of

$$ A - I = \begin{bmatrix} -0.7 & c \\ 0.7 & -c \end{bmatrix} \quad \text{so} \quad x_1 = \begin{bmatrix} c \\ -0.7 \end{bmatrix}, $$

$$ A - (0.3 - c)I = \begin{bmatrix} c & 0 \\ 0.7 & 0.7 \end{bmatrix} \quad \text{so} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. $$

$A$ is not diagonalizable when its eigenvalues are equal: $1 = 0.3 - c$ or $c = -0.7$. (The two eigenvectors above become dependent at $c = -0.7$)

(b) $A^n = SAS^{-1} = S \begin{bmatrix} 1 & 0 \\ 0 & (0.3 - c)^n \end{bmatrix} S^{-1}$

This approaches a limit if $|0.3 - c| < 1$. You could write that out as $-0.7 < c < 1.3$ (Small note: at $c = -0.7$ the eigenvalues are 1 and 1, at $c = 1.3$ the eigenvalues are 1 and $-1$.)
(c) The eigenvectors are in $S$. As $n \to \infty$ the smaller eigenvalue $\lambda_2^n$ goes to zero, leaving

$$A^\infty = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} c & 1 \\ .7 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .7 & -c \end{bmatrix} / (c + .7)$$

$$= \begin{bmatrix} c \\ .7 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} / (c + .7) = \begin{bmatrix} c & c \\ .7 & .7 \end{bmatrix} / (c + .7)$$
3 (30 pts.) Suppose $A$ (3 by 4) has the Singular Value Decomposition (with real orthogonal matrices $U$ and $V$)

$$A = U\Sigma V^T = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^T.$$

(a) Find the rank of $A$ and a basis for its column space $C(A)$.

(b) What are the eigenvalues and eigenvectors of $A^TA$? (You could first multiply $A^T$ times $A$.)

(c) What is $Av_1$? You could start with $V^Tv_1$ and then multiply by $\Sigma$ and $U$ to get $U\Sigma V^Tv_1$.

Solution:

(a) Rank = 2 = rank($A^TA$) = # of nonzero singular values. The vectors $u_1$ and $u_2$ (very sorry about the typo) are a basis for the column space of $A$.

(b) $A^TA = (V\Sigma^TU^T)(U\Sigma V^T) = V\Sigma^T\Sigma V^T$. The eigenvalues of $A^TA$ are 4, 1, 0, 0 in the diagonal matrix $\Sigma^T\Sigma$. The eigenvectors are $v_1, v_2, v_3, v_4$ in the matrix $V$.

(c) $V^Tv_1 = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \\ v_4^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ by orthogonality of the $v$’s.

Multiply by $\Sigma$ to get $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Then multiply by $U$ to get the final answer $2u_1$.

Thus $Av_1 = 2u_1$, which was a main point of the SVD.