Your name is: ________________________________

Please circle your recitation:

1) M2  2-131  I. Ben-Yaacov  2-101  3-3299  pezz
2) M3  2-131  I. Ben-Yaacov  2-101  3-3299  pezz
3) M3  2-132  A. Oblomkov  2-092  3-6228  obloomkov
4) T11  2-132  A. Oblomkov  2-092  3-6228  obloomkov
5) T12  2-132  I. Pak  2-390  3-4390  pak
6) T1  2-131  B. Santoro  2-085  2-1192  bsantoro
7) T1  2-132  I. Pak  2-390  3-4390  pak
8) T2  2-132  B. Santoro  2-085  2-1192  bsantoro
9) T2  2-131  J. Santos  2-180  3-4350  jsantos
We are given two vectors $a$ and $b$ in $\mathbb{R}^4$:

\[
\begin{align*}
a &= \begin{bmatrix} 2 \\ 5 \\ 2 \\ 4 \end{bmatrix} & b &= \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}
\end{align*}
\]

(a) Find the projection $p$ of the vector $b$ onto the line through $a$. Check (!) that the error $e = b - p$ is perpendicular to (what?)

(b) The subspace $S$ of all vectors in $\mathbb{R}^4$ that are perpendicular to this $a$ is 3-dimensional. Find the projection $q$ of $b$ onto this perpendicular subspace $S$. The numerical answer (it doesn’t need a big computation!) is $q =$ ______.
2 (30 pts.) Suppose $q_1, q_2, q_3$ are 3 orthonormal vectors in $\mathbb{R}^n$. They go in the columns of an $n$ by 3 matrix $Q$.

(a) What inequality do you know for $n$?
Is there any condition on $n$ for $Q^TQ = I$ (3 by 3)?
Is there any condition on $n$ for $QQ^T = I$ (n by n)?

(b) Give a nice matrix formula involving $b$ and $Q$, for the projection $p$ of a vector $b$ onto the column space of $Q$.

Complete this sentence: $p$ is the closest vector $\ldots$

(c) Suppose the projection of $b$ onto that column space is $p = c_1q_1 + c_2q_2 + c_3q_3$. Find a formula for $c_1$ that only involves $b$ and $q_1$. (You could take dot products with $q_1$.)
3 (20 pts.) Suppose the 4 by 4 matrix $M$ has four equal rows all containing $a, b, c, d$. We know that $\det(M) = 0$. The problem is to find by any method

$$\det(I + M) = \begin{vmatrix} 1 + a & b & c & d \\ a & 1 + b & c & d \\ a & b & 1 + c & d \\ a & b & c & 1 + d \end{vmatrix}.$$ 

Note If you can’t find $\det(I + M)$ in general, partial credit for the determinant when $a = b = c = d = 1.$
4 (30 pts.) We are looking for the parabola \( y = C + D t + E t^2 \) that gives the least squares fit to these four measurements:

\[
y_1 = 1 \text{ at } t_1 = -2, \ y_2 = 1 \text{ at } t_2 = -1, \ y_3 = 1 \text{ at } t_3 = 1, \ y_4 = 0 \text{ at } t_4 = 2.
\]

(a) Write down the four equations (not solvable!) for the parabola \( C + D t + E t^2 \) to go through those four points. This is the system \( A x = b \) to solve by least squares:

\[
A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = b.
\]

What equations would you solve to find the best \( C, D, E \)?

(b) Compute \( A^T A \). Compute its determinant. Compute its inverse. NOT ASKING FOR \( C, D, E \).

(c) The first two columns of \( A \) are already orthogonal. From column 3, subtract its projection onto the plane of the first two columns. That produces what third orthogonal vector \( v \)? Normalize \( v \) to find the third orthonormal vector \( q_3 \) from Gram-Schmidt.