Your name is: ________________________

Please circle your recitation:

1) M2 2-131 I. Ben-Yaacov 2-101 3-3299 pezz
2) M3 2-131 I. Ben-Yaacov 2-101 3-3299 pezz
3) M3 2-132 A. Oblomkov 2-092 3-6228 oblomkov
4) T11 2-132 A. Oblomkov 2-092 3-6228 oblomkov
5) T12 2-132 I. Pak 2-390 3-4390 pak
6) T1 2-131 B. Santoro 2-085 2-1192 bsantoro
7) T1 2-132 I. Pak 2-390 3-4390 pak
8) T2 2-132 B. Santoro 2-085 2-1192 bsantoro
9) T2 2-131 J. Santos 2-180 3-4350 jsantos

Problems 1–8 are 12 points each; Problem 9 is 4 points.
Thank you for taking 18.06!
Suppose $A$ is an $m$ by $n$ matrix of rank $r$. You multiply it by any $m$ by $n$ invertible matrix $E$ to get $B = EA$.

(a) Circle if true and cross out if false (three parts):

$A$ and $B$ have the

\[
\begin{align*}
\text{same nullspace} \\
\text{same column space} \\
\text{same bases for row space}.
\end{align*}
\]

(b) Suppose the right $E$ gives the row-reduced echelon matrix

\[
EA = R = \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(1) Find a basis for the nullspace of $A$.

(2) True statement: *The nullspace of a matrix is a vector space.*

What does it mean for a set of vectors to be a vector space?

(c) What is the nullspace of a 5 by 4 matrix with linearly independent columns?

What is the nullspace of a 4 by 5 matrix with linearly independent columns?
This matrix $A$ has column 1 + column 2 = column 3:

$$A = \begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 2 \\
0 & 1 & 1 \\
\end{bmatrix}$$

(a) Describe the column space $C(A)$ in two ways:

(1) Give a basis for $C(A)$.

(2) Find all vectors that are perpendicular to $C(A)$.

(b) The projection matrix $P$ onto the column space does not come from the usual formula $A(A^TA)^{-1}A^T$. Why not—what goes wrong with this formula?

(c) Find that matrix $P$ for projection onto the column space of $A$. 
Suppose $P$ is the 3 by 3 projection matrix (so $P = P^T = P^2$) onto the plane $2x + 2y - z = 0$. You do not have to compute this matrix $P$ but you can if you want.

(a) What is the rank of $P$? What are its three eigenvalues? What is its column space?

(b) Is $P$ diagonalizable—why or why not? Find a nonzero vector in its nullspace.

(c) If $b$ is any unit vector in $\mathbb{R}^3$, find the number $q$. Explain your thinking in 1 sentence and 1 equation:

$$q = \|Pb\|^2 + \|b - Pb\|^2.$$
(a) If \( a \neq c \), find the eigenvalue matrix \( \Lambda \) and eigenvector matrix \( S \) in

\[
A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = S \Lambda S^{-1}.
\]

(b) Find the four entries in the matrix \( A^{1000} \).
(a) Suppose $A^T Ax = 0$. This tells us that $Ax$ is in the ___ space of $A^T$. Always $Ax$ is in the ___ space of $A$. Why can you conclude that $Ax = 0$?

(b) Supposing again that $A^T Ax = 0$ we immediately get $x^T A^T Ax = 0$.

From this, show directly that $Ax = 0$.

Every matrix $A^T A$ is symmetric and ________.

(c) The rectangular $m$ by $n$ matrix $A$ always has the same nullspace as the square matrix $A^T A$ (this is proved above). Now deduce that $A$ and $A^T A$ have the same rank.
Suppose $A = \text{ones}(3, 5)$ and $A^T = \text{ones}(5, 3)$ are the 3 by 5 and 5 by 3 matrices of all 1’s.

(a) Find the trace of $AA^T$ and the trace of $A^TA$.

(b) Find the eigenvalues of $AA^T$ and the eigenvalues of $A^TA$.

(c) What is the matrix $\Sigma$ in the singular value decomposition $A = U\Sigma V^T$?
(a) By elimination or otherwise, find the determinant of $A$:

$$A = \begin{bmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \\ v_1 & v_2 & v_3 & 0 \end{bmatrix}$$

(b) If that zero in the lower right corner of $A$ changes to 100, what is the change (if any) in the determinant of $A$? (You can consider its cofactors)

(c) If $(u_1, u_2, u_3)$ is the same as $(v_1, v_2, v_3)$ so $A$ is symmetric, decide if $A$ is or is not positive definite—and why?

(d) Show that this block matrix $M$ is singular for any $u$ and $v$ in $\mathbb{R}^n$, by finding a vector in its nullspace:

$$M = \begin{bmatrix} I & u \\ v^T & v^T u \end{bmatrix}.$$
Suppose $q_1, q_2, q_3$ are orthonormal vectors in $\mathbb{R}^4$ (not $\mathbb{R}^3$!).

(a) What is the length of the vector $v = 2q_1 - 3q_2 + 2q_3$?

(b) What four vectors does Gram-Schmidt produce when it orthonormalizes the vectors $q_1, q_2, q_3, u$?

(c) If $u$ in part (b) is the vector $v$ in part (a), why does Gram-Schmidt break down?

Find a nonzero vector in the nullspace of the 4 by 4 matrix

$$A = \begin{bmatrix} q_1 & q_2 & q_3 & v \end{bmatrix}$$

with columns $q_1, q_2, q_3, v$. 
9 (4 points) PROVE (give a clear reason): If $A$ is a symmetric invertible matrix then $A^{-1}$ is also symmetric.