1. (13 pts.)

(a) 
\[
A = \begin{bmatrix}
0 & 0 & 2 & -2 & 1 & 2 \\
3 & 6 & 0 & 9 & 0 & 3 \\
1 & 2 & 0 & 3 & 1 & 3 \\
-1 & -2 & 2 & -5 & 0 & -1 \\
\end{bmatrix}.
\]

Permuting rows 1 and 2, we get:

\[
\begin{bmatrix}
3 & 6 & 0 & 9 & 0 & 3 \\
0 & 0 & 2 & -2 & 1 & 2 \\
1 & 2 & 0 & 3 & 1 & 3 \\
-1 & -2 & 2 & -5 & 0 & -1 \\
\end{bmatrix}.
\]

Now we can eliminate entries (3, 1) and (4, 1) to get:

\[
\begin{bmatrix}
3 & 6 & 0 & 9 & 0 & 3 \\
0 & 0 & 2 & -2 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 2 & -2 & 0 & 0 \\
\end{bmatrix}.
\]

The second pivot is now element (2, 3), and this pivot can be used to eliminate element (4, 3):

\[
\begin{bmatrix}
3 & 6 & 0 & 9 & 0 & 3 \\
0 & 0 & 2 & -2 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & -1 & -2 \\
\end{bmatrix}.
\]

The next pivot is element (3, 5), and it allows to eliminate element (4, 5):

\[
\begin{bmatrix}
3 & 6 & 0 & 9 & 0 & 3 \\
0 & 0 & 2 & -2 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

The matrix now is in echelon form. To get the reduced row echelon form, we first scale row 1 by $1/3$ and row 2 by $1/2$:

\[
\begin{bmatrix}
1 & 2 & 0 & 3 & 0 & 1 \\
0 & 0 & 1 & -1 & 1/2 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

We still need to eliminate entry (2, 5) (as $x_5$ is a pivot variable) and this is done by subtracting $1/2$ of row 3 from row 2:

\[
R = \begin{bmatrix}
1 & 2 & 0 & 3 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]
and this is the reduced row echelon form.

(b) The rank of $A$ is 3 since we found 3 pivot variables: $x_1, x_3$ and $x_5$.

(c) If we take $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ and we redo the eliminations on the augmented matrix $[A|b]$, we get that $Ax = b$ is equivalent to $Ex = d$ where $d = \begin{bmatrix} b_2/3 \\ b_1/2 - b_3/2 + b_2/6 \\ b_3 - b_2/3 \\ b_4 - b_1 + b_3 \end{bmatrix}$. If we take $b$ such that $b_4 - b_1 + b_3 \neq 0$ then $Ax = b$ has no solution.

(d) When doing the elimination with $b = \begin{bmatrix} 22 \\ 24 \\ 16 \\ 6 \end{bmatrix}$, we get (see previous subquestion) $d = \begin{bmatrix} 8 \\ 7 \\ 8 \\ 0 \end{bmatrix}$. Thus a particular solution is

$$x_p = \begin{bmatrix} 8 \\ 0 \\ 7 \\ 8 \\ 0 \end{bmatrix}.$$

To get all solutions, we need to add linear combinations of the special solutions of the nullspace. We have a special solution for each free variable $x_2, x_4$ and $x_6$. All solutions to $Ax = b$ are thus given by:

$$\begin{bmatrix} 8 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} = + \begin{bmatrix} 8 - 2x_2 - 3x_4 - x_6 \\ x_2 \\ 7 + x_4 \\ x_4 \\ 8 - 2x_6 \\ x_6 \end{bmatrix}.$$

(e) No, since the nullspace contains non-zero vectors.

(g) The rank of $A^T A$ is also 3. Indeed let us prove that the rank of $A^T A$ is always equal to the rank of $A$ (without doing any eliminations).

To see this, we first show that $N(A) = N(A^T A)$. It is clear that any $x$ with $A x = 0$ satisfies $A^T A x = 0$. The converse is also true: If $A^T A x = 0$, observe that for $w = A x$ we have that $w \in N(A^T)$ and $w = C(A)$ which implies that $w = 0$ as $N(A^T) \cap C(A) = \{0\}$. In other words $A^T A x = 0$ implies that $A x = 0$. The fact that $N(A) = N(A^T A)$ now implies that the dimensions of these subspaces are the same and thus we have $\text{rank}(A) = \text{rank}(A^T A)$.

2. (6 pts.) Consider the space $F$ spanned by the 4 vectors $v_1 = (4, 2, 4, 2)$, $v_2 = (-1, 4, 5, 10)$, $v_3 = (-5, 2, 1, 8)$ and $v_4 = (6, 6, 10, 10)$.

(a) The $v_i$’s are not linearly independent. Indeed, if you consider the matrix

$$
A = \begin{bmatrix}
4 & -1 & -5 & 6 \\
2 & 4 & 2 & 6 \\
4 & 5 & 1 & 10 \\
2 & 10 & 8 & 10 \\
\end{bmatrix},
$$

and do eliminations, we’ll get only two pivots. The matrix $A$ would need to have a nullspace of dimension 0 for the vectors to be linearly independent.

(b) $v_1$ and $v_2$ forms a basis of $F$. Any two of the $v_i$’s would work here as none of them is a multiple of another.

(c) The dimension of $F$ is 2 as we have two pivots.

(d) $v_1 + 2v_2 + 3v_3$, $v_1 - v_2$ and $v_4$ cannot be linearly independent since 3 vectors of a subspace of dimension 2 are never linearly independent.

3. (5 pts.) Consider the subspace $F$ of all $3 \times 3$ symmetric matrices with zeroes on the diagonal.

(a) Consider the 3 matrices:

$$
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}.
$$

A linear combination of these matrices gives the matrix:

$$
\begin{bmatrix}
0 & a & b \\
a & 0 & c \\
b & c & 0 \\
\end{bmatrix}.
$$

To get the 0 matrix, we must have $a = b = c = 0$ implying that the 3 matrices are linearly independent. Furthermore we can get any symmetric matrix with zeroes on the diagonal by choosing $a, b$ and $c$ appropriately, and thus these 3 vectors span the subspace. Hence they form a basis.

(b) We’ll need $1 + 2 + \cdots + n - 1$ matrices in the basis, for a total of $\frac{n(n-1)}{2}$. 

3
4. (4 pts.) Suppose we couldn’t find an index \( l \). This means that \( v_1, v_2, \ldots, v_{k-1}, v_k, v_l \) are linearly dependent for every \( l = k + 1, \ldots, n \). Since \( v_1, \ldots, v_k \) are linearly independent, it means that \( v_l \) linearly depends on \( v_1, \ldots, v_k \) for \( l > k \). This implies that any vector which is a linear combination of all the \( v_i \)'s can be expressed as a linear combination of just \( v_1, \ldots, v_k \). In other words, \( v_1, \ldots, v_k \) form a basis of \( C(A) \) and this contradicts the fact that the rank (and thus the dimension of \( C(A) \)) is greater than \( k \).

5. (12 pts.) Exercise 14 of section 3.6 on page 181. \( A = BC \) where \( B \) is invertible (since it is lower triangular with nonzeros on the diagonal).

- \( N(A) \). The nullspace \( N(A) \) is equal to \( N(C) \) (since \( B \) is invertible: \( BCx = 0 \) if and only if \( Cx = 0 \)). As \( C \) is in echelon form and \( x_4 \) is a free variable, we can just take that special solution as the only vector in the basis of \( N(C) = N(A) \):

\[
\begin{bmatrix}
0 \\
1 \\
-2 \\
1
\end{bmatrix}
\]

- \( R(A) \). Similarly \( R(A) = R(C) \) (from \( y = A^T u = C^T B^T u = C^T (B^T u) \) and \( B^T \) being invertible). We can just take all 3 row vectors of \( C \) as basis:

\[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
2 \\
3
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1 \\
2
\end{bmatrix}
\]

Thus the rank of \( A \) is 3.

- \( C(A) \). As the rank of \( A \) and thus the dimension of \( C(A) \) is 3, we have that \( C(A) \) is all of \( R^3 \). Thus we can take any basis of \( R^3 \), say the 3 unit vectors.

- \( N(A^T) \). As \( \text{dim}(C(A)) + \text{dim}(N(A^T)) = 3 \), we have that \( \text{dim}(N(A^T)) = 0 \) and thus a basis of \( N(A^T) \) contains 0 vectors (not the 0 vector).