1. (13 pts.)

Consider the differential equation \( \frac{du}{dt} = Au \) where \( u \) is 2-dimensional and

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]

(a) The characteristic polynomial of \( A \) is

\[
\det \begin{bmatrix}
-\lambda & 1 \\
-1 & -\lambda
\end{bmatrix} = \lambda^2 + 1,
\]

and therefore the eigenvalues are \( \lambda_1 = i \) and \( \lambda_2 = -i \). The eigenvectors are \( v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \) (or any complex multiple of it) and \( v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} \).

(b) Since \( A \) has distinct eigenvalues, it is diagonalizable and we have that \( A = V \Lambda V^{-1} \) where

\[
V = \begin{bmatrix}
-i & i \\
1 & 1
\end{bmatrix},
\]

and

\[
\Lambda = \begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix}.
\]

The inverse of \( V \) is:

\[
V^{-1} = \frac{1}{2} \begin{bmatrix}
i & 1 \\
-i & 1
\end{bmatrix}.
\]

(Observe that since the columns of \( V \) are orthogonal but not of unit norm, \( V^{-1} \) is almost \( V^H \); we could have scaled \( V \) by \( \frac{1}{\sqrt{2}} \) to make its inverse equal to its Hermitian.)

Now, as \( A \) is diagonalizable, we can compute \( e^{At} \) by \( Ve^{\Lambda t}V^{-1} \):

\[
e^{At} = \frac{1}{2} \begin{bmatrix}
-i & i \\
1 & 1
\end{bmatrix} \begin{bmatrix}
e^{it} & 0 \\
0 & e^{-it}
\end{bmatrix} \begin{bmatrix}
i & 1 \\
-i & 1
\end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix}
-i & i \\
1 & 1
\end{bmatrix} \begin{bmatrix}
e^{it} & 0 \\
0 & e^{-it}
\end{bmatrix} \begin{bmatrix}
i & 1 \\
-i & 1
\end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix}
-ie^{it} & ie^{-it} \\
e^{it} & e^{-it}
\end{bmatrix} \begin{bmatrix}
i & 1 \\
-i & 1
\end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix}
e^{it} + e^{-it} & -ie^{it} + ie^{-it} \\
ie^{it} - ie^{-it} & e^{it} + e^{-it}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{bmatrix}.
\]
(c) Assuming \( u(0) = \begin{bmatrix} 9 \\ 2 \end{bmatrix} \) we get
\[
u(t) = e^{At} u(0) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 9\cos(t) + 2\sin(t) \\ -9\sin(t) + 2\cos(t) \end{bmatrix}.
\]

(d) The differential equation is not stable (since the eigenvalues have their real part equal to 0, and not strictly less than 0).

(e) It will be a circle, as \( u_1^2(t) + u_2^2(t) = 9^2 + 2^2 = 40. \)

2. (10 pts.) Let
\[
A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(a) Since \( A \) is upper triangular, the eigenvalues are just the diagonal elements (as \( \det(A - \lambda I) \) is the product of the diagonal elements of \( A - \lambda I \)). Thus all eigenvalues are 0.

(b) The nullspace of \( A \) has dimension 1, so we have only 1 linearly independent eigenvector.

(c) To compute \( e^{tA} \), we have to use the infinite series \( e^{tA} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \cdots \). The powers of \( A \) are:
\[
A^2 = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]
\[
A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]
\[
A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

and all powers \( A^k \) with \( k \geq 4 \) are equal to 0. Thus,
\[
e^{tA} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 \]
\[
= I + t \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{t^3}{6} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 1 & t & 2t + \frac{t^2}{2} & 3t + 2t^2 + \frac{t^3}{6} \\ 0 & 1 & t & 2t + \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]
3. (13 pts.) Let
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}.
\]
(a) To compute an \( LDL^T \) factorization of \( A \), we need to do eliminations to transform \( A \) into an upper triangular matrix. Pivoting on \((1,1)\), we get:
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & -1 & -5 \\
0 & -5 & -7
\end{bmatrix}.
\]
Pivoting on \((2,2)\), we get:
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & -1 & -5 \\
0 & 0 & 18
\end{bmatrix}.
\]
This is equal to \( DU \), where
\[
U = \begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 5 \\
0 & 0 & 1
\end{bmatrix},
\]
and
\[
D = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 18
\end{bmatrix}.
\]
Now, since \( A \) is symmetric, we know that the \( LDU \) factorization will be \( LDL^T \), and thus we have
\[
L = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & 5 & 1
\end{bmatrix}.
\]
(b) As all the row sums are 6, one of the eigenvalue is 6 (say \( \lambda_1 \)), with corresponding eigenvector
\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]
After scaling we get that the eigenvector is:
\[
v_1 = \begin{bmatrix}
1\sqrt{3} \\
1/\sqrt{3} \\
1/\sqrt{3}
\end{bmatrix}.
\]
To get the others, we could compute the characteristic polynomial. But let’s try to compute them otherwise. The trace of \( A \) is 6, so the sum of the other 2 eigenvalues is 0, and we have \( \lambda_2 = -\lambda_3 \). The product of the eigenvalues is the determinant of \( A \), which equals to \(-18\) (either we could compute it explicitly or remember that it is the product of the pivots after elimination (modulus a possible sign change if we have done permutations)). Thus, \( \lambda_2 \lambda_3 = -3 \) and we get \( \lambda_2 = \sqrt{3} \) and \( \lambda_3 = -\sqrt{3} \).
Let’s now get an eigenvector for $\lambda_2$. We have:

$$A - \lambda_2 I = \begin{bmatrix} 1 - \sqrt{3} & 2 & 3 \\ 2 & 3 - \sqrt{3} & 1 \\ 3 & 1 & 2 - \sqrt{3} \end{bmatrix}.$$ 

We need to find the nullspace, so we perform eliminations. Adding the first row times $(1 + \sqrt{3})$ to the second, we get:

$$\begin{bmatrix} 1 - \sqrt{3} & 2 & 3 \\ 5 + \sqrt{3} & 4 + 3\sqrt{3} & 3 \\ * & * & * \end{bmatrix}.$$ 

We don’t care about the 3rd row, as it will become 0 after further eliminations. The special solution here is

$$\begin{bmatrix} \frac{\sqrt{3} - 1}{2} \\ -\frac{1 + \sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}.$$ 

We can scale it by $\sqrt{3}$ to get an eigenvector of unit norm:

$$v_2 = \begin{bmatrix} \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} - \frac{1}{2} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$ 

We do it similarly for $\lambda_3$. In fact, we can easily guess $v_3$ now as it is orthogonal to both $v_1$ and $v_2$ (since $A$ is symmetric). We get:

$$v_3 = \begin{bmatrix} -\frac{1}{2\sqrt{3}} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$ 

(c) We just have to take for $Q$ the matrix whose columns are $v_1, v_2, v_3$ and for $\Lambda$ the diagonal matrix of eigenvalues.

$$\Lambda = \begin{bmatrix} 6 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \end{bmatrix},$$

and

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2} - \frac{1}{2\sqrt{3}} & \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} - \frac{1}{2} & \frac{1}{\sqrt{3}} & \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{2} - \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$ 

(d) In both cases we get -18, and this is the determinant of $A$; the determinant of $A$ is both the product of the pivots and of the eigenvalues.

4. (4 pts.) Since $A$ is symmetric then $A = tI$ is also symmetric. Furthermore, for any eigenvalue $\lambda_i$ of $A$, we have that $\lambda_i + t$ is an eigenvalue of $A + tI$ and vice versa (since $(A + tI)v = \lambda v + tv$ for $v$ an eigenvalue of $A$ corresponding to $\lambda$). So, if $t > -\min_i \lambda_i$ then $A + tI$ has all its (real) eigenvalues greater than 0, and thus is positive definite.