LECTURE 1
INTRODUCTION/BASIC CONVEXITY CONCEPTS

LECTURE OUTLINE

• Convex Optimization Problems
• Why is Convexity Important in Optimization
• Multipliers and Lagrangian Duality
• Min Common/Max Crossing Duality
• Convex sets and functions
• Epigraphs
• Closed convex functions
• Recognizing convex functions
OPTIMIZATION PROBLEMS

• Generic form:

\[
\begin{align*}
\text{minimize} \; & f(x) \\
\text{subject to} \; & x \in C
\end{align*}
\]

Cost function \( f : \mathbb{R}^n \mapsto \mathbb{R} \), constraint set \( C \), e.g.,

\[
C = X \cap \{ x \mid h_1(x) = 0, \ldots, h_m(x) = 0 \} \cap \{ x \mid g_1(x) \leq 0, \ldots, g_r(x) \leq 0 \}
\]

• Examples of problem classifications:
  – Continuous vs discrete
  – Linear vs nonlinear
  – Deterministic vs stochastic
  – Static vs dynamic

• Convex programming problems are those for which \( f \) is convex and \( C \) is convex (they are continuous problems).

• However, convexity permeates all of optimization, including discrete problems.
WHY IS CONVEXITY SO SPECIAL?

• A convex function has no local minima that are not global
• A convex set has a nonempty relative interior
• A convex set is connected and has feasible directions at any point
• A nonconvex function can be “convexified” while maintaining the optimality of its global minima
• The existence of a global minimum of a convex function over a convex set is conveniently characterized in terms of directions of recession
• A polyhedral convex set is characterized in terms of a finite set of extreme points and extreme directions
• A real-valued convex function is continuous and has nice differentiability properties
• Closed convex cones are self-dual with respect to polarity
• Convex, lower semicontinuous functions are self-dual with respect to conjugacy
CONVEXITY AND DUALITY

- Consider the (primal) problem

\[
\text{minimize } f(x) \quad \text{s.t. } g_1(x) \leq 0, \ldots, g_r(x) \leq 0
\]

- We introduce multiplier vectors \( \mu = (\mu_1, \ldots, \mu_r) \geq 0 \) and form the Lagrangian function

\[
L(x, \mu) = f(x) + \sum_{j=1}^{r} \mu_j g_j(x), \quad x \in \mathbb{R}^n, \mu \in \mathbb{R}^r.
\]

- Dual function

\[
q(\mu) = \inf_{x \in \mathbb{R}^n} L(x, \mu)
\]

- Dual problem: Maximize \( q(\mu) \) over \( \mu \geq 0 \)

- Motivation: Under favorable circumstances (strong duality) the optimal values of the primal and dual problems are equal, and their optimal solutions are related
KEY DUALITY RELATIONS

- **Optimal primal value**

\[ f^* = \inf_{g_j(x) \leq 0, j=1,\ldots,r} f(x) = \inf_{x \in \mathbb{R}^n} \sup_{\mu \geq 0} L(x, \mu) \]

- **Optimal dual value**

\[ q^* = \sup_{\mu \geq 0} q(\mu) = \sup_{\mu \geq 0} \inf_{x \in \mathbb{R}^n} L(x, \mu) \]

- We always have \( q^* \leq f^* \) (weak duality - important in discrete optimization problems).

- Under favorable circumstances (convexity in the primal problem, plus \( \ldots \)):
  - We have \( q^* = f^* \) (strong duality)
  - If \( \mu^* \) is optimal dual solution, all optimal primal solutions minimize \( L(x, \mu^*) \)

- This opens a wealth of analytical and computational possibilities, and insightful interpretations.

- Note that the equality of “\( \sup \inf \)” and “\( \inf \sup \)” is a key issue in minimax theory and game theory.
MIN COMMON/MAX CROSSING DUALITY

- All of duality theory and all of (convex/concave) minimax theory can be developed/explained in terms of this one figure.

- This is the novel aspect of the treatment (although the ideas are closely connected to conjugate convex function theory)

- The machinery of convex analysis is needed to flesh out this figure, and to rule out the exceptional/pathological behavior shown in (c).
EXCEPTIONAL BEHAVIOR

- If convex structure is so favorable, what is the source of exceptional/pathological behavior [like in (c) of the preceding slide]?

- **Answer:** Some common operations on convex sets do not preserve some basic properties.

- **Example:** A linearly transformed closed convex set need not be closed (contrary to compact and polyhedral sets).

- This is a major reason for the analytical difficulties in convex analysis and pathological behavior in convex optimization (and the favorable character of polyhedral sets).
COURSE OUTLINE

1) **Basic Convexity Concepts (2):** Convex sets and functions. Convex and affine hulls. Closure, relative interior, and continuity.


3) **Convex Optimization Concepts (1):** Existence of optimal solutions. Partial minimization. Saddle point and minimax theory.

4) **Min common/max crossing duality (1):** MC/MC duality. Special cases in constrained minimization and minimax. Strong duality theorem. Existence of dual optimal solutions.

5) **Duality applications (2):** Constrained optimization (Lagrangian, Fenchel, and conic duality). Subdifferential theory and optimality conditions. Minimax theorems. Nonconvex problems and estimates of the duality gap.
WHAT TO EXPECT FROM THIS COURSE

• We aim:
  – To develop insight and deep understanding of a fundamental optimization topic
  – To treat rigorously an important branch of applied math, and to provide some appreciation of the research in the field

• Mathematical level:
  – Prerequisites are linear algebra (preferably abstract) and real analysis (a course in each)
  – Proofs are important ... but the rich geometry helps guide the mathematics

• We will make maximum use of visualization and figures

• Applications: They are many and pervasive ... but don’t expect much in this course. The book by Boyd and Vandenberghe describes a lot of practical convex optimization models (http://www.stanford.edu/boyd/cvxbook.html)

• Handouts: Slides, 1st chapter, material in http://www.athenasc.com/convexity.html
A NOTE ON THESE SLIDES

- These slides are a teaching aid, not a text
- Don’t expect strict mathematical rigor
- The statements of theorems are fairly precise, but the proofs are not
- Many proofs have been omitted or greatly abbreviated
- Figures are meant to convey and enhance ideas, not to express them precisely
- The omitted proofs and a much fuller discussion can be found in the “Convex Optimization” textbook and handouts
SOME MATH CONVENTIONS

• All of our work is done in $\mathbb{R}^n$: space of $n$-tuples $x = (x_1, \ldots, x_n)$

• All vectors are assumed column vectors

• “$'$” denotes transpose, so we use $x'$ to denote a row vector

• $x'y$ is the inner product $\sum_{i=1}^{n} x_i y_i$ of vectors $x$ and $y$

• $\|x\| = \sqrt{x'x}$ is the (Euclidean) norm of $x$. We use this norm almost exclusively

• See the appendix for an overview of the linear algebra and real analysis background that we will use
A subset $C$ of $\mathbb{R}^n$ is called convex if

$$\alpha x + (1 - \alpha)y \in C, \quad \forall x, y \in C, \forall \alpha \in [0, 1]$$

Operations that preserve convexity

- Intersection, scalar multiplication, vector sum, closure, interior, linear transformations

Cones: Sets $C$ such that $\lambda x \in C$ for all $\lambda > 0$ and $x \in C$ (not always convex or closed)
• Let $C$ be a convex subset of $\mathbb{R}^n$. A function $f : C \mapsto \mathbb{R}$ is called convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in C$, and $\alpha \in [0, 1]$.

• If $f$ is a convex function, then all its level sets \{ $x \in C \mid f(x) \leq a$ \} and \{ $x \in C \mid f(x) < a$ \}, where $a$ is a scalar, are convex.
EXTENDED REAL-VALUED FUNCTIONS

- The epigraph of a function $f : X \mapsto [-\infty, \infty]$ is the subset of $\mathbb{R}^{n+1}$ given by

$$\text{epi}(f) = \{(x, w) \mid x \in X, w \in \mathbb{R}, f(x) \leq w\}$$

- The effective domain of $f$ is the set

$$\text{dom}(f) = \{x \in X \mid f(x) < \infty\}$$

- We say that $f$ is proper if $f(x) < \infty$ for at least one $x \in X$ and $f(x) > -\infty$ for all $x \in X$, and we will call $f$ improper if it is not proper.

- Note that $f$ is proper if and only if its epigraph is nonempty and does not contain a “vertical line.”

- An extended real-valued function $f : X \mapsto [-\infty, \infty]$ is called lower semicontinuous at a vector $x \in X$ if $f(x) \leq \lim \inf_{k \to \infty} f(x_k)$ for every sequence $\{x_k\} \subset X$ with $x_k \to x$.

- We say that $f$ is closed if $\text{epi}(f)$ is a closed set.
CLOSEDNESS AND SEMICONTINUITY

• Proposition: For a function $f : \mathbb{R}^n \mapsto [-\infty, \infty]$, the following are equivalent:

(i) $\{ x \mid f(x) \leq a \}$ is closed for every scalar $a$.
(ii) $f$ is lower semicontinuous at all $x \in \mathbb{R}^n$.
(iii) $f$ is closed.

• Note that:
  - If $f$ is lower semicontinuous at all $x \in \text{dom}(f)$, it is not necessarily closed
  - If $f$ is closed, $\text{dom}(f)$ is not necessarily closed

• Proposition: Let $f : X \mapsto [-\infty, \infty]$ be a function. If $\text{dom}(f)$ is closed and $f$ is lower semicontinuous at all $x \in \text{dom}(f)$, then $f$ is closed.
Let $C$ be a convex subset of $\mathbb{R}^n$. An extended real-valued function $f : C \mapsto [-\infty, \infty]$ is called convex if $\text{epi}(f)$ is a convex subset of $\mathbb{R}^{n+1}$.

If $f$ is proper, this definition is equivalent to

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in C$, and $\alpha \in [0, 1]$.

An improper closed convex function is very peculiar: it takes an infinite value ($\infty$ or $-\infty$) at every point.
RECOGNIZING CONVEX FUNCTIONS

- Some important classes of elementary convex functions: Affine functions, positive semidefinite quadratic functions, norm functions, etc.

- Proposition: Let $f_i : \mathbb{R}^n \mapsto (-\infty, \infty]$, $i \in I$, be given functions ($I$ is an arbitrary index set).
  (a) The function $g : \mathbb{R}^n \mapsto (-\infty, \infty]$ given by

  $$g(x) = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x), \quad \lambda_i > 0$$

  is convex (or closed) if $f_1, \ldots, f_m$ are convex (respectively, closed).

  (b) The function $g : \mathbb{R}^n \mapsto (-\infty, \infty]$ given by

  $$g(x) = f(Ax)$$

  where $A$ is an $m \times n$ matrix is convex (or closed) if $f$ is convex (respectively, closed).

  (c) The function $g : \mathbb{R}^n \mapsto (-\infty, \infty]$ given by

  $$g(x) = \sup_{i \in I} f_i(x)$$

  is convex (or closed) if the $f_i$ are convex (respectively, closed).
LECTURE 2

LECTURE OUTLINE

• Differentiable Convex Functions
• Convex and Affine Hulls
• Caratheodory’s Theorem
• Closure, Relative Interior, Continuity
DIFFERENTIABLE CONVEX FUNCTIONS

Let $C \subset \mathbb{R}^n$ be a convex set and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable over $\mathbb{R}^n$.

(a) The function $f$ is convex over $C$ iff

$$f(z) \geq f(x) + (z - x)'\nabla f(x), \quad \forall \, x, z \in C$$

Implies that $x^*$ minimizes $f$ over $C$ iff

$$\nabla f(x^*)'(x - x^*) \geq 0, \quad \forall \, x \in C$$

(b) If the inequality is strict whenever $x \neq z$, then $f$ is strictly convex over $C$, i.e., for all $\alpha \in (0, 1)$ and $x, y \in C$, with $x \neq y$

$$f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y)$$
PROOF IDEAS

\[ f(z) + (x - z)\nabla f(z) = \alpha f(x) + (1 - \alpha) f(y) \]

\[ f(z) + (y - z)\nabla f(z) = f(z) + \left(\frac{f(x + \alpha(z - x)) - f(x)}{\alpha}\right) \]

(a)

(b)
Let $C$ be a convex subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice continuously differentiable over $\mathbb{R}^n$.

(a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then $f$ is convex over $C$.

(b) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then $f$ is strictly convex over $C$.

(c) If $C$ is open and $f$ is convex over $C$, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

**Proof:** (a) By mean value theorem, for $x, y \in C$

$$f(y) = f(x) + (y-x)' \nabla f(x) + \frac{1}{2} (y-x)' \nabla^2 f(x+\alpha(y-x))(y-x)$$

for some $\alpha \in [0, 1]$. Using the positive semidefiniteness of $\nabla^2 f$, we obtain

$$f(y) \geq f(x) + (y-x)' \nabla f(x), \quad \forall x, y \in C$$

From the preceding result, $f$ is convex.

(b) Similar to (a), we have $f(y) > f(x) + (y-x)' \nabla f(x)$ for all $x, y \in C$ with $x \neq y$, and we use the preceding result.
CONVEX AND AFFINE HULLS

• Given a set $X \subset \mathbb{R}^n$:

• A convex combination of elements of $X$ is a vector of the form $\sum_{i=1}^{m} \alpha_i x_i$, where $x_i \in X$, $\alpha_i \geq 0$, and $\sum_{i=1}^{m} \alpha_i = 1$.

• The convex hull of $X$, denoted $\text{conv}(X)$, is the intersection of all convex sets containing $X$ (also the set of all convex combinations from $X$).

• The affine hull of $X$, denoted $\text{aff}(X)$, is the intersection of all affine sets containing $X$ (an affine set is a set of the form $\bar{x} + S$, where $S$ is a subspace). Note that $\text{aff}(X)$ is itself an affine set.

• A nonnegative combination of elements of $X$ is a vector of the form $\sum_{i=1}^{m} \alpha_i x_i$, where $x_i \in X$ and $\alpha_i \geq 0$ for all $i$.

• The cone generated by $X$, denoted $\text{cone}(X)$, is the set of all nonnegative combinations from $X$:
  
  – It is a convex cone containing the origin.
  
  – It need not be closed (even if $X$ is compact).
  
  – If $X$ is a finite set, $\text{cone}(X)$ is closed (non-trivial to show!)
• Let $X$ be a nonempty subset of $\mathbb{R}^n$.

(a) Every $x \neq 0$ in $\text{cone}(X)$ can be represented as a positive combination of vectors $x_1, \ldots, x_m$ from $X$ that are linearly independent (so $m \leq n$).

(b) Every $x \not\in X$ that belongs to $\text{conv}(X)$ can be represented as a convex combination of at most $n + 1$ vectors.
PROOF OF CARATHEODORY’S THEOREM

(a) Let \( x \) be a nonzero vector in \( \text{cone}(X) \), and let \( m \) be the smallest integer such that \( x \) has the form \( \sum_{i=1}^{m} \alpha_i x_i \), where \( \alpha_i > 0 \) and \( x_i \in X \) for all \( i = 1, \ldots, m \). If the vectors \( x_i \) were linearly dependent, there would exist \( \lambda_1, \ldots, \lambda_m \), with

\[
\sum_{i=1}^{m} \lambda_i x_i = 0
\]

and at least one of the \( \lambda_i \) is positive. Consider

\[
\sum_{i=1}^{m} (\alpha_i - \gamma \lambda_i) x_i,
\]

where \( \gamma \) is the largest \( \gamma \) such that \( \alpha_i - \gamma \lambda_i \geq 0 \) for all \( i \). This combination provides a representation of \( x \) as a positive combination of fewer than \( m \) vectors of \( X \) – a contradiction. Therefore, \( x_1, \ldots, x_m \), are linearly independent.

(b) Apply part (a) to the subset of \( \mathbb{R}^{n+1} \)

\[
Y = \{(z, 1) \mid z \in X\}
\]

consider \( \text{cone}(Y) \), represent \( (x, 1) \in \text{cone}(Y) \) in terms of at most \( n + 1 \) vectors, etc.
AN APPLICATION OF CARATHEODORY

• The convex hull of a compact set is compact.

**Proof:** Let $X$ be compact. We take a sequence in $\text{conv}(X)$ and show that it has a convergent subsequence whose limit is in $\text{conv}(X)$.

By Caratheodory, a sequence in $\text{conv}(X)$ can be expressed as $\left\{ \sum_{i=1}^{n+1} \alpha_i^k x_i^k \right\}$, where for all $k$ and $i$, $\alpha_i^k \geq 0$, $x_i^k \in X$, and $\sum_{i=1}^{n+1} \alpha_i^k = 1$. Since

$$\left\{ (\alpha_1^k, \ldots, \alpha_{n+1}^k, x_1^k, \ldots, x_{n+1}^k) \right\}$$

is bounded, it has a limit point

$$\left\{ (\alpha_1, \ldots, \alpha_{n+1}, x_1, \ldots, x_{n+1}) \right\},$$

which must satisfy $\sum_{i=1}^{n+1} \alpha_i = 1$, and $\alpha_i \geq 0$, $x_i \in X$ for all $i$. Thus, the vector $\sum_{i=1}^{n+1} \alpha_i x_i$, which belongs to $\text{conv}(X)$, is a limit point of the sequence $\left\{ \sum_{i=1}^{n+1} \alpha_i^k x_i^k \right\}$, so $\text{conv}(X)$ is compact.

Q.E.D.

• Note the convex hull of a closed set need not be closed.
**RELATIVE INTERIOR**

• *x* is a *relative interior point* of *C*, if *x* is an interior point of *C* relative to aff(*C*).

• ri(*C*) denotes the *relative interior of* *C*, i.e., the set of all relative interior points of *C*.

• **Line Segment Principle:** If *C* is a convex set, *x* ∈ ri(*C*) and *x* ∈ cl(*C*), then all points on the line segment connecting *x* and *x*, except possibly *x*, belong to ri(*C*).
ADDITIONAL MAJOR RESULTS

- Let $C$ be a nonempty convex set.
  
  (a) $\text{ri}(C)$ is a nonempty convex set, and has the same affine hull as $C$.
  
  (b) $x \in \text{ri}(C)$ if and only if every line segment in $C$ having $x$ as one endpoint can be prolonged beyond $x$ without leaving $C$.

\[X = \left\{ \sum_{i=1}^{m} \alpha_i z_i \mid \sum_{i=1}^{m} \alpha_i < 1, \alpha_i > 0, i = 1, \ldots, m \right\}\]

Proof: (a) Assume that $0 \in C$. We choose $m$ linearly independent vectors $z_1, \ldots, z_m \in C$, where $m$ is the dimension of $\text{aff}(C)$, and we let

(b) $\Rightarrow$ is clear by the def. of rel. interior. Reverse: argue by contradiction; take any $\overline{x} \in \text{ri}(C)$; use prolongation assumption and Line Segment Princ.
OPTIMIZATION APPLICATION

• A concave function $f : \mathbb{R}^n \mapsto \mathbb{R}$ that attains its minimum over a convex set $X$ at an $x^* \in \text{ri}(X)$ must be constant over $X$.

\[
\text{Proof: (By contradiction.) Let } x \in X \text{ be such that } f(x) > f(x^*). \text{ Prolong beyond } x^* \text{ the line segment } x\text{-to-}x^* \text{ to a point } \overline{x} \in X. \text{ By concavity of } f, \text{ we have for some } \alpha \in (0, 1) \\

f(x^*) \geq \alpha f(x) + (1 - \alpha) f(\overline{x}),
\]

and since $f(x) > f(x^*)$, we must have $f(x^*) > f(\overline{x})$ - a contradiction. Q.E.D.

• Corollary: A linear function can attain a minimum only at the boundary of a convex set.
• The relative interior of a convex set is equal to the relative interior of its closure.

• The closure of the relative interior of a convex set is equal to its closure.

• Relative interior and closure commute with Cartesian product and inverse image under a linear transformation.

• Relative interior commutes with image under a linear transformation and vector sum, but closure does not.

• Neither relative interior nor closure commute with set intersection.
CLOSURE VS RELATIVE INTERIOR

• Let $C$ be a nonempty convex set. Then $\text{ri}(C)$ and $\text{cl}(C)$ are “not too different for each other.”

• Proposition:

  (a) We have $\text{cl}(C) = \text{cl}(\text{ri}(C))$.

  (b) We have $\text{ri}(C) = \text{ri}(\text{cl}(C))$.

  (c) Let $\overline{C}$ be another nonempty convex set. Then the following three conditions are equivalent:

     (i) $C$ and $\overline{C}$ have the same rel. interior.

     (ii) $C$ and $\overline{C}$ have the same closure.

     (iii) $\text{ri}(C) \subset \overline{C} \subset \text{cl}(C)$.

Proof: (a) Since $\text{ri}(C) \subset C$, we have $\text{cl}(\text{ri}(C)) \subset \text{cl}(C)$. Conversely, let $\overline{x} \in \text{cl}(C)$. Let $x \in \text{ri}(C)$. By the Line Segment Principle, we have $\alpha x + (1 - \alpha)\overline{x} \in \text{ri}(C)$ for all $\alpha \in (0, 1]$. Thus, $\overline{x}$ is the limit of a sequence that lies in $\text{ri}(C)$, so $\overline{x} \in \text{cl}(\text{ri}(C))$. 

\[ \text{Diagram: } \]
LINEAR TRANSFORMATIONS

- Let $C$ be a nonempty convex subset of $\mathbb{R}^n$ and let $A$ be an $m \times n$ matrix.

(a) We have $A \cdot \text{ri}(C) = \text{ri}(A \cdot C)$.

(b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$. Furthermore, if $C$ is bounded, then $A \cdot \text{cl}(C) = \text{cl}(A \cdot C)$.

**Proof:** (a) Intuition: Spheres within $C$ are mapped onto spheres within $A \cdot C$ (relative to the affine hull).

(b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$, since if a sequence $\{x_k\} \subset C$ converges to some $x \in \text{cl}(C)$ then the sequence $\{Ax_k\}$, which belongs to $A \cdot C$, converges to $Ax$, implying that $Ax \in \text{cl}(A \cdot C)$.

To show the converse, assuming that $C$ is bounded, choose any $z \in \text{cl}(A \cdot C)$. Then, there exists a sequence $\{x_k\} \subset C$ such that $Ax_k \rightarrow z$. Since $C$ is bounded, $\{x_k\}$ has a subsequence that converges to some $x \in \text{cl}(C)$, and we must have $Ax = z$. It follows that $z \in A \cdot \text{cl}(C)$. **Q.E.D.**

Note that in general, we may have

$$A \cdot \text{int}(C) \neq \text{int}(A \cdot C), \quad A \cdot \text{cl}(C) \neq \text{cl}(A \cdot C)$$
INTERSECTIONS AND VECTOR SUMS

- Let $C_1$ and $C_2$ be nonempty convex sets.

(a) We have

$$\text{ri}(C_1 + C_2) = \text{ri}(C_1) + \text{ri}(C_2),$$

$$\text{cl}(C_1) + \text{cl}(C_2) \subset \text{cl}(C_1 + C_2)$$

If one of $C_1$ and $C_2$ is bounded, then

$$\text{cl}(C_1) + \text{cl}(C_2) = \text{cl}(C_1 + C_2)$$

(b) If $\text{ri}(C_1) \cap \text{ri}(C_2) \neq \emptyset$, then

$$\text{ri}(C_1 \cap C_2) = \text{ri}(C_1) \cap \text{ri}(C_2),$$

$$\text{cl}(C_1 \cap C_2) = \text{cl}(C_1) \cap \text{cl}(C_2)$$

Proof of (a): $C_1 + C_2$ is the result of the linear transformation $(x_1, x_2) \mapsto x_1 + x_2$.

- Counterexample for (b):

$$C_1 = \{x \mid x \leq 0\}, \quad C_2 = \{x \mid x \geq 0\}$$
CONTINUITY OF CONVEX FUNCTIONS

- If $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex, then it is continuous.

**Proof:** We will show that $f$ is continuous at 0. By convexity, $f$ is bounded within the unit cube by the maximum value of $f$ over the corners of the cube.

Consider sequence $x_k \to 0$ and the sequences $y_k = x_k / \|x_k\|_\infty$, $z_k = -x_k / \|x_k\|_\infty$. Then

$$f(x_k) \leq (1 - \|x_k\|_\infty) f(0) + \|x_k\|_\infty f(y_k)$$

$$f(0) \leq \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) + \frac{1}{\|x_k\|_\infty + 1} f(x_k)$$

Since $\|x_k\|_\infty \to 0$, $f(x_k) \to f(0)$. **Q.E.D.**

- Extension to continuity over $\text{ri}(\text{dom}(f))$. 
CLOSURES OF FUNCTIONS

• The closure of a function \( f : X \mapsto [-\infty, \infty] \) is the function \( \text{cl} f : \mathbb{R}^n \mapsto [-\infty, \infty] \) with

\[
\text{epi}(\text{cl} f) = \text{cl}(\text{epi}(f))
\]

• The convex closure of \( f \) is the function \( \tilde{\text{cl}} f \) with

\[
\text{epi}(\tilde{\text{cl}} f) = \text{cl}(\text{conv}(\text{epi}(f)))
\]

• Proposition: For any \( f : X \mapsto [-\infty, \infty] \)

\[
\inf_{x \in X} f(x) = \inf_{x \in \mathbb{R}^n} (\text{cl} f)(x) = \inf_{x \in \mathbb{R}^n} (\tilde{\text{cl}} f)(x).
\]

Also, any vector that attains the infimum of \( f \) over \( X \) also attains the infimum of \( \text{cl} f \) and \( \tilde{\text{cl}} f \).

• Proposition: For any \( f : X \mapsto [-\infty, \infty] \):

  (a) \( \text{cl} f \) (\( \tilde{\text{cl}} f \)) is the greatest closed (closed convex, resp.) function majorized by \( f \).

  (b) If \( f \) is convex, then \( \text{cl} f \) is convex, and it is proper if and only if \( f \) is proper. Also,

\[
(\text{cl} f)(x) = f(x), \quad \forall x \in \text{ri}(\text{dom}(f)),
\]

and if \( x \in \text{ri}(\text{dom}(f)) \) and \( y \in \text{dom}(\text{cl} f) \),

\[
(\text{cl} f)(y) = \lim_{\alpha \downarrow 0} f(y + \alpha(x - y)).
\]
LECTURE 3

LECTURE OUTLINE

• Recession cones
• Directions of recession of convex functions
• Nonemptiness of closed set intersections
• Linear and Quadratic Programming
• Preservation of closure under linear transformation
RECESSION CONE OF A CONVEX SET

- Given a nonempty convex set $C$, a vector $d$ is a direction of recession if starting at any $x$ in $C$ and going indefinitely along $d$, we never cross the relative boundary of $C$ to points outside $C$:

$$x + \alpha d \in C, \quad \forall x \in C, \quad \forall \alpha \geq 0$$

- **Recession cone** of $C$ (denoted by $R_C$): The set of all directions of recession.

- $R_C$ is a cone containing the origin.
RECESSION CONE THEOREM

• Let $C$ be a nonempty closed convex set.

(a) The recession cone $R_C$ is a closed convex cone.

(b) A vector $d$ belongs to $R_C$ if and only if there exists a vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.

(c) $R_C$ contains a nonzero direction if and only if $C$ is unbounded.

(d) The recession cones of $C$ and $\text{ri}(C)$ are equal.

(e) If $D$ is another closed convex set such that $C \cap D \neq \emptyset$, we have

$$R_{C \cap D} = R_C \cap R_D$$

More generally, for any collection of closed convex sets $C_i, i \in I$, where $I$ is an arbitrary index set and $\cap_{i \in I} C_i$ is nonempty, we have

$$R_{\cap_{i \in I} C_i} = \cap_{i \in I} R_{C_i}$$
• Let $d \neq 0$ be such that there exists a vector $x \in C$ with $x + \alpha d \in C$ for all $\alpha \geq 0$. We fix $\bar{x} \in C$ and $\alpha > 0$, and we show that $\bar{x} + \alpha d \in C$. By scaling $d$, it is enough to show that $\bar{x} + d \in C$.

Let $z_k = x + kd$ for $k = 1, 2, \ldots$, and $d_k = (z_k - \bar{x})\|d\|/\|z_k - \bar{x}\|$. We have

$$
\frac{d_k}{\|d\|} = \frac{\|z_k - x\|}{\|z_k - \bar{x}\| \|d\|} \frac{d}{\|d\|} + \frac{x - \bar{x}}{\|z_k - \bar{x}\|}, \quad \|z_k - x\| \to 1, \quad \frac{x - \bar{x}}{\|z_k - \bar{x}\|} \to 0,
$$

so $d_k \to d$ and $\bar{x} + d_k \to \bar{x} + d$. Use the convexity and closedness of $C$ to conclude that $\bar{x} + d \in C$. 
LINEALITY SPACE

• The lineality space of a convex set $C$, denoted by $L_C$, is the subspace of vectors $d$ such that $d \in R_C$ and $-d \in R_C$:

\[ L_C = R_C \cap (-R_C) \]

• If $d \in L_C$, the entire line defined by $d$ is contained in $C$, starting at any point of $C$.

• Decomposition of a Convex Set: Let $C$ be a nonempty convex subset of $\mathbb{R}^n$. Then,

\[ C = L_C + (C \cap L_C^\perp). \]

• True also if $L_C$ is replaced by a subset $S \subset L_C$. 
DIRECTIONS OF RECESSION OF A FUNCTION

- Some basic geometric observations:
  - The “horizontal directions” in the recession cone of the epigraph of a convex function \( f \) are directions along which the level sets are unbounded.
  - Along these directions the level sets \( \{x \mid f(x) \leq \gamma\} \) are unbounded and \( f \) is monotonically nondecreasing.

- These are the directions of recession of \( f \).
RECESSION CONE OF LEVEL SETS

• Proposition: Let $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be a closed proper convex function and consider the level sets $V_\gamma = \{ x \mid f(x) \leq \gamma \}$, where $\gamma$ is a scalar. Then:

(a) All the nonempty level sets $V_\gamma$ have the same recession cone, given by

$$R_{V_\gamma} = \{ d \mid (d, 0) \in R_{\text{epi}(f)} \}$$

(b) If one nonempty level set $V_\gamma$ is compact, then all nonempty level sets are compact.

Proof: For each fixed $\gamma$ for which $V_\gamma$ is nonempty,

$$\{(x, \gamma) \mid x \in V_\gamma\} = \text{epi}(f) \cap \{(x, \gamma) \mid x \in \mathbb{R}^n\}$$

The recession cone of the set on the left is $\{(d, 0) \mid d \in R_{V_\gamma}\}$. The recession cone of the set on the right is the intersection of $R_{\text{epi}(f)}$ and the recession cone of $\{(x, \gamma) \mid x \in \mathbb{R}^n\}$. Thus we have

$$\{(d, 0) \mid d \in R_{V_\gamma}\} = \{(d, 0) \mid (d, 0) \in R_{\text{epi}(f)}\},$$

from which the result follows.
RECESSION CONE OF A CONVEX FUNCTION

- For a closed proper convex function $f : \mathbb{R}^n \mapsto (-\infty, \infty]$, the (common) recession cone of the nonempty level sets $V_\gamma = \{x \mid f(x) \leq \gamma\}, \gamma \in \mathbb{R}$, is the recession cone of $f$, and is denoted by $R_f$.

- Terminology:
  - $d \in R_f$: a direction of recession of $f$.
  - $L_f = R_f \cap (-R_f)$: the lineality space of $f$.
  - $d \in L_f$: a direction of constancy of $f$.

- Example: For the pos. semidefinite quadratic

  $$f(x) = x'Qx + a'x + b,$$

  the recession cone and constancy space are

  $$R_f = \{d \mid Qd = 0, a'd \leq 0\}, \quad L_f = \{d \mid Qd = 0, a'd = 0\}$$
RECESSION FUNCTION

• Function \( r_f : \mathbb{R}^n \mapsto (-\infty, \infty] \) whose epigraph is \( \text{Re} \) : the recession function of \( f \).

• Characterizes the recession cone:

\[
R_f = \{ d \mid r_f(d) \leq 0 \}, \quad L_f = \{ d \mid r_f(d) = r_f(-d) = 0 \}
\]

• Can be shown that

\[
\begin{align*}
 r_f(d) &= \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} = \lim_{\alpha \to \infty} \frac{f(x + \alpha d) - f(x)}{\alpha} \\
\end{align*}
\]

• Thus \( r_f(d) \) is the “asymptotic slope” of \( f \) in the direction \( d \). In fact,

\[
 r_f(d) = \lim_{\alpha \to \infty} \nabla f(x + \alpha d)'d, \quad \forall \ x, d \in \mathbb{R}^n
\]

if \( f \) is differentiable.

• Calculus of recession functions:

\[
r_{f_1 + \cdots + f_m}(d) = r_{f_1}(d) + \cdots + r_{f_m}(d)
\]

\[
r_{\sup_{i \in I} f_i}(d) = \sup_{i \in I} r_{f_i}(d)
\]
• \( y \) is a direction of recession in (a)-(d).
• This behavior is independent of the starting point \( x \), as long as \( x \in \text{dom}(f) \).
THE ROLE OF CLOSED SET INTERSECTIONS

• A fundamental question: Given a sequence of nonempty closed sets \( \{C_k\} \) in \( \mathbb{R}^n \) with \( C_{k+1} \subset C_k \) for all \( k \), when is \( \bigcap_{k=0}^{\infty} C_k \) nonempty?

• Set intersection theorems are significant in at least three major contexts, which we will discuss in what follows:

1. Does a function \( f : \mathbb{R}^n \mapsto (-\infty, \infty] \) attain a minimum over a set \( X \)? This is true iff the intersection of the nonempty level sets \( \{x \in X \mid f(x) \leq \gamma_k\} \) is nonempty.

2. If \( C \) is closed and \( A \) is a matrix, is \( AC \) closed? Special case:
   - If \( C_1 \) and \( C_2 \) are closed, is \( C_1 + C_2 \) closed?

3. If \( F(x, z) \) is closed, is \( f(x) = \inf_z F(x, z) \) closed? (Critical question in duality theory.) Can be addressed by using the relation

\[
P(\text{epi}(F)) \subset \text{epi}(f) \subset \text{cl}\left(P(\text{epi}(F'))\right)
\]

where \( P(\cdot) \) is projection on the space of \( (x, w) \).
**ASYMPTOTIC DIRECTIONS**

- Given nested sequence \( \{C_k\} \) of closed convex sets, \( \{x_k\} \) is an *asymptotic sequence* if
  \[
  x_k \in C_k, \quad x_k \neq 0, \quad k = 0, 1, \ldots
  \]
  \[
  \|x_k\| \to \infty, \quad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}
  \]
  where \( d \) is a nonzero common direction of recession of the sets \( C_k \).

- As a special case we define asymptotic sequence of a closed convex set \( C \) (use \( C_k \equiv C \)).

- Every unbounded \( \{x_k\} \) with \( x_k \in C_k \) has an asymptotic subsequence.

- \( \{x_k\} \) is called *retractive* if for some \( \bar{k} \), we have
  \[
  x_k - d \in C_k, \quad \forall \ k \geq \bar{k}.
  \]
RETRACTIVE SEQUENCES

- A nested sequence \( \{C_k\} \) of closed convex sets is *retractive* if all its asymptotic sequences are retractive.

- Intersections and Cartesian products of retractive set sequences are retractive.

- A closed halfspace (viewed as a sequence with identical components) is retractive.

- A polyhedral set is retractive. Also the vector sum of a convex compact set and a retractive convex set is retractive.

- Nonpolyhedral cones and level sets of quadratic functions need not be retractive.
SET INTERSECTION THEOREM I

**Proposition:** If \( \{C_k\} \) is retractive, then \( \bigcap_{k=0}^{\infty} C_k \) is nonempty.

- **Key proof ideas:**
  
  (a) The intersection \( \bigcap_{k=0}^{\infty} C_k \) is empty iff the sequence \( \{x_k\} \) of minimum norm vectors of \( C_k \) is unbounded (so a subsequence is asymptotic).

  (b) An asymptotic sequence \( \{x_k\} \) of minimum norm vectors cannot be retractive, because such a sequence eventually gets closer to 0 when shifted opposite to the asymptotic direction.
SET INTERSECTION THEOREM II

**Proposition:** Let \( \{C_k\} \) be a nested sequence of nonempty closed convex sets, and \( X \) be a retractive set such that all the sets \( \overline{C}_k = X \cap C_k \) are nonempty. Assume that

\[
R_X \cap R \subset L,
\]

where

\[
R = \bigcap_{k=0}^{\infty} R_{C_k}, \quad L = \bigcap_{k=0}^{\infty} L_{C_k}
\]

Then \( \{\overline{C}_k\} \) is retractive and \( \bigcap_{k=0}^{\infty} \overline{C}_k \) is nonempty.

- **Special case:** \( X = \mathbb{R}^n, R = L \).

**Proof:** The set of common directions of recession of \( C_k \) is \( R_X \cap R \). For any asymptotic sequence \( \{x_k\} \) corresponding to \( d \in R_X \cap R \):

1. \( x_k - d \in C_k \) (because \( d \in L \))
2. \( x_k - d \in X \) (because \( X \) is retractive)

So \( \{\overline{C}_k\} \) is retractive.
Consider \( \bigcap_{k=0}^{\infty} \overline{C}_k \), with \( \overline{C}_k = X \cap C_k \)

- The condition \( R_X \cap R \subset L \) holds
- In the figure on the left, \( X \) is polyhedral.
- In the figure on the right, \( X \) is nonpolyhedral and nonretractive, and

\[ \bigcap_{k=0}^{\infty} \overline{C}_k = \emptyset \]
Theorem: Let

\[ f(x) = x'Qx + c'x, \quad X = \{ x \mid a_j'x + b_j \leq 0, \ j = 1, \ldots, r \}, \]

where \( Q \) is symmetric positive semidefinite. If the minimal value of \( f \) over \( X \) is finite, there exists a minimum of \( f \) over \( X \).

Proof: (Outline) Write

\[
\text{Set of Minima} = X \cap \{ x \mid x'Qx + c'x \leq \gamma_k \}
\]

with

\[ \gamma_k \downarrow f^* = \inf_{x \in X} f(x). \]

Verify the condition \( R_X \cap R \subset L \) of the preceding set intersection theorem, where \( R \) and \( L \) are the sets of common recession and lineality directions of the sets

\[
\{ x \mid x'Qx + c'x \leq \gamma_k \}
\]

Q.E.D.
CLOSURE UNDER LINEAR TRANSFORMATIONS

- Let $C$ be a nonempty closed convex, and let $A$ be a matrix with nullspace $N(A)$.
  
  (a) $AC$ is closed if $R_C \cap N(A) \subset L_C$.

  (b) $A(X \cap C)$ is closed if $X$ is a retractive set and
  
  $R_X \cap R_C \cap N(A) \subset L_C$,

Proof: (Outline) Let $\{y_k\} \subset AC$ with $y_k \to \bar{y}$. We prove $\cap_{k=0}^{\infty} C_k \neq \emptyset$, where $C_k = C \cap N_k$, and

$$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \{z \mid \|z - \bar{y}\| \leq \|y_k - \bar{y}\|\}$$

- Special Case: $AX$ is closed if $X$ is polyhedral.
NEED TO ASSUME THAT $X$ IS RETRACTIVE

Consider closedness of $A(X \cap C)$

- In both examples the condition

$$R_X \cap R_C \cap N(A) \subset L_C$$

is satisfied.
- However, in the example on the right, $X$ is not retractive, and the set $A(X \cap C)$ is not closed.
LECTURE 4

LECTURE OUTLINE

• Hyperplane separation
• Proper separation
• Nonvertical hyperplanes
• Convex conjugate functions
• Conjugacy theorem
• Examples
A hyperplane is a set of the form \( \{ x \mid a'x = b \} \), where \( a \) is nonzero vector in \( \mathbb{R}^n \) and \( b \) is a scalar.

We say that two sets \( C_1 \) and \( C_2 \) are separated by a hyperplane \( H = \{ x \mid a'x = b \} \) if each lies in a different closed halfspace associated with \( H \), i.e.,

either \( a'x_1 \leq b \leq a'x_2, \quad \forall x_1 \in C_1, \; \forall x_2 \in C_2, \)

or \( a'x_2 \leq b \leq a'x_1, \quad \forall x_1 \in C_1, \; \forall x_2 \in C_2 \)

If \( \bar{x} \) belongs to the closure of a set \( C \), a hyperplane that separates \( C \) and the singleton set \( \{ \bar{x} \} \) is said be supporting \( C \) at \( \bar{x} \).
VISUALIZATION

- Separating and supporting hyperplanes:

\[ a \] (a) \[ C_1 \] \[ a \] (b) \[ C \]

- A separating \( \{ x \mid a'x = b \} \) that is disjoint from \( C_1 \) and \( C_2 \) is called strictly separating:

\[ a'x_1 < b < a'x_2, \quad \forall x_1 \in C_1, \forall x_2 \in C_2 \]
**SUPPORTING HYPERPLANE THEOREM**

- Let $C$ be convex and let $\bar{x}$ be a vector that is not an interior point of $C$. Then, there exists a hyperplane that passes through $\bar{x}$ and contains $C$ in one of its closed halfspaces.

**Proof:** Take a sequence $\{x_k\}$ that does not belong to $\text{cl}(C)$ and converges to $\bar{x}$. Let $\hat{x}_k$ be the projection of $x_k$ on $\text{cl}(C)$. We have for all $x \in \text{cl}(C')$

$$a'_k x \geq a'_k x_k, \quad \forall \ x \in \text{cl}(C'), \ \forall \ k = 0, 1, \ldots,$$

where $a_k = (\hat{x}_k - x_k)/\|\hat{x}_k - x_k\|$. Let $a$ be a limit point of $\{a_k\}$, and take limit as $k \to \infty$. Q.E.D.
SEPARATING HYPERPLANE THEOREM

Let $C_1$ and $C_2$ be two nonempty convex subsets of $\mathbb{R}^n$. If $C_1$ and $C_2$ are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \quad \forall x_1 \in C_1, \forall x_2 \in C_2.$$ 

Proof: Consider the convex set

$$C_1 - C_2 = \{x_2 - x_1 \mid x_1 \in C_1, x_2 \in C_2\}$$

Since $C_1$ and $C_2$ are disjoint, the origin does not belong to $C_1 - C_2$, so by the Supporting Hyperplane Theorem, there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \quad \forall x \in C_1 - C_2,$$

which is equivalent to the desired relation. Q.E.D.
• **Strict Separation Theorem:** Let $C_1$ and $C_2$ be two disjoint nonempty convex sets. If $C_1$ is closed, and $C_2$ is compact, there exists a hyperplane that strictly separates them.

![Diagram of Strict Separation Theorem](image)

**Proof:** (Outline) Consider the set $C_1 - C_2$. Since $C_1$ is closed and $C_2$ is compact, $C_1 - C_2$ is closed. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Let $\overline{x}_1 - \overline{x}_2$ be the projection of 0 onto $C_1 - C_2$. The strictly separating hyperplane is constructed as in (b).

• **Note:** Any conditions that guarantee closedness of $C_1 - C_2$ guarantee existence of a strictly separating hyperplane. However, there may exist a strictly separating hyperplane without $C_1 - C_2$ being closed.
• **Fundamental Characterization:** The closure of the convex hull of a set $C \subset \mathbb{R}^n$ is the intersection of the closed halfspaces that contain $C$. (Proof uses the strict separation theorem.)

• We say that a hyperplane properly separates $C_1$ and $C_2$ if it separates $C_1$ and $C_2$ and does not fully contain both $C_1$ and $C_2$.

![Diagram Illustrating Proper Separation](a) (b) (c)

• **Proper Separation Theorem:** Let $C_1$ and $C_2$ be two nonempty convex subsets of $\mathbb{R}^n$. There exists a hyperplane that properly separates $C_1$ and $C_2$ if and only if

$$\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset$$
PROPER POLYHEDRAL SEPARATION

• Recall that two convex sets $C$ and $P$ such that

$$\text{ri}(C) \cap \text{ri}(P) = \emptyset$$

can be properly separated, i.e., by a hyperplane that does not contain both $C$ and $P$.

• If $P$ is polyhedral and the slightly stronger condition

$$\text{ri}(C) \cap P = \emptyset$$

holds, then the properly separating hyperplane can be chosen so that it does not contain the non-polyhedral set $C$ while it may contain $P$.

On the left, the separating hyperplane can be chosen so that it does not contain $C$. On the right where $P$ is not polyhedral, this is not possible.
NONVERTICAL HYPERPLANES

- A hyperplane in $\mathbb{R}^{n+1}$ with normal $(\mu, \beta)$ is nonvertical if $\beta \neq 0$.

- It intersects the $(n+1)$st axis at $\xi = (\mu/\beta)'\overline{u} + \overline{w}$, where $(\overline{u}, \overline{w})$ is any vector on the hyperplane.

- A nonvertical hyperplane that contains the epigraph of a function in its “upper” halfspace, provides lower bounds to the function values.

- The epigraph of a proper convex function does not contain a vertical line, so it appears plausible that it is contained in the “upper” halfspace of some nonvertical hyperplane.
NONVERTICAL HYPERPLANE THEOREM

- Let $C$ be a nonempty convex subset of $\mathbb{R}^{n+1}$ that contains no vertical lines. Then:

  (a) $C$ is contained in a closed halfspace of a non-vertical hyperplane, i.e., there exist $\mu \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ with $\beta \neq 0$, and $\gamma \in \mathbb{R}$ such that $\mu'u + \beta w \geq \gamma$ for all $(u, w) \in C$.

  (b) If $(\overline{u}, \overline{w}) \notin \text{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(\overline{u}, \overline{w})$ and $C$.

**Proof:** Note that $\text{cl}(C)$ contains no vert. line [since $C$ contains no vert. line, $\text{ri}(C)$ contains no vert. line, and $\text{ri}(C)$ and $\text{cl}(C)$ have the same recession cone]. So we just consider the case: $C$ closed.

(a) $C$ is the intersection of the closed halfspaces containing $C$. If all these corresponded to vertical hyperplanes, $C$ would contain a vertical line.

(b) There is a hyperplane strictly separating $(\overline{u}, \overline{w})$ and $C$. If it is nonvertical, we are done, so assume it is vertical. “Add” to this vertical hyperplane a small $\epsilon$-multiple of a nonvertical hyperplane containing $C$ in one of its halfspaces as per (a).
CONJUGATE CONVEX FUNCTIONS

• Consider a function $f$ and its epigraph

Nonvertical hyperplanes supporting $\text{epi}(f)$

$\mapsto$ Crossing points of vertical axis

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x' y - f(x) \}, \quad y \in \mathbb{R}^n. \]

• For any $f : \mathbb{R}^n \mapsto [-\infty, \infty]$, its conjugate convex function is defined by

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x' y - f(x) \}, \quad y \in \mathbb{R}^n \]
EXAMPLES

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \left\{ x'y - f(x) \right\}, \quad y \in \mathbb{R}^n \]
CONJUGATE OF CONJUGATE

• From the definition

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x'y - f(x) \}, \quad y \in \mathbb{R}^n, \]

note that \( h \) is convex and closed.

• Reason: \( \text{epi}(f^*) \) is the intersection of the epigraphs of the linear functions of \( y \)

\[ x'y - f(x) \]

as \( x \) ranges over \( \mathbb{R}^n \).

• Consider the conjugate of the conjugate:

\[ f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y'x - f^*(y) \}, \quad x \in \mathbb{R}^n. \]

• \( f^{**} \) is convex and closed.

• Important fact/Conjugacy theorem: If \( f \) is closed proper convex, then \( f^{**} = f \).
CONJUGACY THEOREM - VISUALIZATION

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x'y - f(x) \}, \quad y \in \mathbb{R}^n \]

\[ f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y'x - f^*(y) \}, \quad x \in \mathbb{R}^n \]

- If \( f \) is closed convex proper, then \( f^{**} = f \).
CONJUGACY THEOREM

- Let $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be a function, let $\tilde{c}l\ f$ be its convex closure, let $f^*$ be its convex conjugate, and consider the conjugate of $f^*$,

$$f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y' x - f^*(y) \}, \quad x \in \mathbb{R}^n$$

(a) We have

$$f(x) \geq f^{**}(x), \quad \forall \ x \in \mathbb{R}^n$$

(b) If $f$ is convex, then properness of any one of $f$, $f^*$, and $f^{**}$ implies properness of the other two.

(c) If $f$ is closed proper and convex, then

$$f(x) = f^{**}(x), \quad \forall \ x \in \mathbb{R}^n$$

(d) If $\tilde{c}l\ f(x) > -\infty$ for all $x \in \mathbb{R}^n$, then

$$\tilde{c}l\ f(x) = f^{**}(x), \quad \forall \ x \in \mathbb{R}^n$$
A COUNTEREXAMPLE

• A counterexample (with closed convex but improper \( f \)) showing the need to assume properness in order for \( f = f^{**} \):

\[
f(x) = \begin{cases} 
\infty & \text{if } x > 0, \\
-\infty & \text{if } x \leq 0.
\end{cases}
\]

We have

\[
f^*(y) = \infty, \quad \forall \ y \in \mathbb{R}^n,
\]

\[
f^{**}(x) = -\infty, \quad \forall \ x \in \mathbb{R}^n.
\]

But

\[
\tilde{\operatorname{cl}} f = f,
\]

so \( \tilde{\operatorname{cl}} f \neq f^{**} \).
A FEW EXAMPLES

- $l_p$ and $l_q$ norm conjugacy, where $\frac{1}{p} + \frac{1}{q} = 1$

\[ f(x) = \frac{1}{p} \sum_{i=1}^{n} |x_i|^p, \quad f^*(y) = \frac{1}{q} \sum_{i=1}^{n} |y_i|^q \]

- Conjugate of a strictly convex quadratic

\[ f(x) = \frac{1}{2} x'Qx + a'x + b, \]
\[ f^*(y) = \frac{1}{2} (y - a)'Q^{-1}(y - a) - b. \]

- Conjugate of a function obtained by invertible linear transformation/translation of a function $p$

\[ f(x) = p(A(x - c)) + a'x + b, \]
\[ f^*(y) = q((A')^{-1}(y - a)) + c'y + d, \]
where $q$ is the conjugate of $p$ and $d = -(c'a + b)$. 
SUPPOR T FUNCTIONS

• Conjugate of indicator function $\delta_X$ of set $X$

$$\sigma_X(y) = \sup_{x \in X} y'x$$

is called the support function of $X$.

• $\text{epi}(\sigma_X)$ is a closed convex cone.

• The sets $X$, $\text{cl}(X)$, $\text{conv}(X)$, and $\text{cl}(\text{conv}(X))$ all have the same support function (by the conjugacy theorem).

• To determine $\sigma_X(y)$ for a given vector $y$, we project the set $X$ on the line determined by $y$, we find $\hat{x}$, the extreme point of projection in the direction $y$, and we scale by setting

$$\sigma_X(y) = \|\hat{x}\| \cdot \|y\|$$