Supplemental Notes

At irregular intervals, I’ll be giving you notes of this form; as in these notes, they will often be for the purposes of correcting my mistakes or finishing things that I had started in class but turned out to be more interesting than the available time allowed.

By the way, in these notes, vectors will be denoted by an overarrow, and unit vectors by a hat (as in \( \vec{E}, \hat{j} \)), primarily to reproduce my personal blackboard notation.

Anyway, our story thus far (Friday, September 8): We had gotten to the point

\[
\vec{E} = \sigma \hat{j} \int_{-L/2}^{L/2} \left[ \int_{-L/2}^{L/2} \frac{dx'}{(x'^2 + r^2 + z'^2)^{3/2}} \right] dz'.
\]

The last thing mentioned was that the first \((x')\) integral could be done by methods used in the prior example (the long charged rod). Better yet, we can use the integral given for Problem 5.11 on Page 202, with \(x'\) instead of \(x\) and \(a^2 = r^2 + z'^2\). Doing the definite integral, we then have

\[
\vec{E} = \sigma r \hat{j} \int_{-L/2}^{L/2} \frac{dz'}{(z'^2 + r^2)(z'^2 + r^2 + (L^2/4))^{1/2}}.
\]

For now, let’s just concentrate on the \(z'\)-integral.

There are many ways to do this integral, of varying degrees of complexity. It is tempting, and it works, to try

\[
z' = r \tan \theta, \quad dz' = r \sec^2 \theta \, d\theta,
\]

which is the substitution made previously (for the first, \(x'\), integral). This gives

\[
\frac{dz'}{z'^2 + r^2} = \frac{d\theta}{r}, \quad \left( z'^2 + r^2 + (L^2/4) \right)^{1/2} = r \left( \sec^2 \theta + (L^2/4r^2) \right)^{1/2},
\]

and so

\[
\int \frac{dz'}{(z'^2 + r^2)(z'^2 + r^2 + (L^2/4))^{1/2}} = \frac{1}{r^2} \int \frac{d\theta}{(\sec^2 \theta + (L^2/4r^2))^{1/2}}.
\]
(Note that the limits have been left off of the integral; believe me, it’s for the best.) Now the tricky part comes in; we have

$$\frac{d\theta}{(\sec^2 \theta + (L^2/4r^2))^{1/2}} = \frac{\cos \theta \, d\theta}{(1 + (L^2/4r^2) \cos^2 \theta)^{1/2}} \quad du = \frac{1}{(L/2r)((4r^2/L^2) + 1 - u^2)^{1/2}},$$

where the substitution $u = \sin \theta$ has been made. Now, let

$$u = \sqrt{(4r^2/L^2) + 1} \sin \phi, \quad du = \sqrt{(4r^2/L^2) + 1} \cos \phi \, d\phi,$$

so the whole thing reduces to

$$\int \frac{dz'}{(z'^2 + r^2)(z'^2 + r^2 + (L^2/4))^{1/2}} = \frac{2}{Lr} \phi = \frac{2}{rL} \sin^{-1} \left( \frac{u}{\sqrt{(4r^2/L^2) + 1}} \right) = \frac{2}{Lr} \sin^{-1} \left( \frac{\sin \left( \tan^{-1}(z/r) \right)}{\sqrt{(4r^2/L^2) + 1}} \right) = \frac{2}{Lr} \sin^{-1} \left( \frac{z/r}{\sqrt{(4r^2/L^2) + 1 \sqrt{1 + (z^2/r^2)}}} \right).$$

Evaluating this between the limits (we’re back to the $z'$ limits now) of $\pm L/2$ gives

$$\frac{4}{Lr} \sin^{-1} \left( \frac{L/2r}{\sqrt{(4r^2/L^2) + 1 \sqrt{1 + (L^2/4r^2)}}} \right) = \frac{4}{Lr} \sin^{-1} \left( \frac{L^2}{L^2 + 4r^2} \right).$$

Combining with the above expression for the first integral, we end up with

$$\vec{E} = 4\sigma \sin^{-1} \left( \frac{L^2}{L^2 + 4r^2} \right) \hat{j}.$$

Whew.

We’re not really done, yet. What we can do is see what happens if $r \gg L$; then, the argument of the inverse sine is approximately $L^2/4r^2$, and so this is approximately the value of the inverse sine, and so the asymptotic form for the electric field is

$$\vec{E} = \frac{4\sigma L^2}{4r^2} \hat{j} = \frac{\sigma L^2}{r^2} \hat{j} = \frac{Q}{r^2} \hat{j},$$
as expected (remembering that $Q = \sigma L^2$).

If $L \gg r$, the argument of the inverse sine is approximately one, and the inverse sine is $\pi/2$; the electric field in this limit is then

$$\vec{E} = 2\pi \sigma \hat{j},$$

a result that we will verify using Gauss’s Law in a much simpler but equivalent manner.

So, this problem serves many purposes. I’m going to claim that the physics needed to set up this problem is not hard. We then reduced the problem to a series of single-variable integrals, one of which was moderately hard and one of which was very hard. As indicated, the moderately hard integral is given in the text (granted, you have to know how to look), and comes up frequently in this class. The very hard integral is more difficult than anything I would intentionally assign; as it is, it’s an 18.023 problem that comes much later in that class. Anyway, after doing all of the work, we get a fairly nice result that reduces to the expected results in the limit as we get far from the square (so that it looks like a point charge) and very close to the square (so that it looks like an infinite plane). It is always a good idea to check to see that this correspondence is achieved.