2.2 Similarity Parameters (from governing equations)

Non-dimensionalize and normalize basic equations by scaling:

Identify characteristic scales for the problem

- velocity \( U \)
- length \( L \)
- time \( T \)
- pressure \( p_{\text{ref}} - p_{\text{ref}} \)

\[
\begin{align*}
\vec{v} &= U \vec{v}^* \\
\vec{x} &= L \vec{x}^* \\
t &= T t^* \\
p &= (p_{\text{ref}} - p_{\text{ref}}) p^*
\end{align*}
\]

All ()* quantities are dimensionless and normalized (i.e. \( O(1) \)), e.g. \( \frac{\partial \vec{v}^*}{\partial x^*} = O(1) \).

Apply to governing equations: (also internal constitution, boundary conditions)

- Continuity (incompressible flow):
  \[
  \nabla \cdot \vec{v} = \frac{U}{L} \nabla^* \cdot \vec{v}^* = 0, \quad \nabla^* \cdot \vec{v}^* = 0
  \]

- Navier-Stokes:
  \[
  \frac{\partial \vec{v}^*}{\partial t^*} + (\vec{v}^* \cdot \nabla^*) \vec{v}^* = -\frac{1}{\rho} \nabla^* p + \nu \nabla^2 \vec{v}^* - g \hat{j}
  \]
  \[
  \frac{U}{T} \frac{\partial \vec{v}^*}{\partial t^*} + \frac{U^2}{L} \left( \vec{v}^* \cdot \nabla^* \right) \vec{v}^* = -\frac{p_{\text{ref}} - p_{\text{ref}}}{\rho L} \nabla^* p^* + \frac{\nu U}{L^2} \nabla^2 \vec{v}^* - g \hat{j}
  \]
  divide through by \( \frac{U^2}{L} \) (order of magnitude of the convective inertia term)
  \[
  \frac{\tilde{L}}{U T} \left( \frac{\partial \vec{v}^*}{\partial t^*} \right)^* + \left( (\vec{v}^* \cdot \nabla^*) \vec{v}^* \right)^* = -\frac{p_{\text{ref}} - p_{\text{ref}}}{\rho U^2} \left( \nabla^* p \right)^* + \frac{\nu}{UL} \left( \nabla^2 \vec{v}^* \right)^* - \frac{g \hat{j}}{U^2}
  \]
Since all \((\cdot)^*\) terms are \(O(1)\), the coefficients \(\sim \frac{\alpha \vec{v}}{\nu} \) measure the relative importance of each term (as compared to the convective inertia term):

- \(\frac{L}{UT} = S = \text{Strouhal number} \sim \frac{\text{Eulerian inertia}}{\text{convective inertia}} \frac{\alpha \vec{v}}{(\vec{v} \cdot \nabla) \vec{v}}\)
  
  is a measure of transient behavior. For example e.g. if \(T \gg \frac{L}{U}, S \ll 1\), ignore \(\frac{\alpha \vec{v}}{\nu} \to \text{assume steady-state.} \)

- \(\frac{p_o-p_v}{\frac{1}{2} \rho U^2} = \sigma = \text{cavitation number} \text{ (measures likelihood of cavitation)}\)
  
  If \(\sigma \gg 1\), no cavitation. Alternatively, when cavitation is not a concern \(p = p_o p^*\).

- \(\frac{1}{2} \frac{p_v}{\rho U^2} = E_u = \text{Euler number} \sim \frac{\text{pressure force}}{\text{inertia force}}\)

- \(\frac{UL}{\nu} = R_e = \text{Reynold's number} \sim \frac{\text{inertia force}}{\text{viscous force}}\)
  
  If \(R_e \gg 1\), ignore viscosity.

- \(\sqrt{\frac{U^2}{gL}} = \frac{U}{\sqrt{gL}} = Fr = \text{Froude number} \sim \left(\frac{\text{inertia force}}{\text{gravity force}}\right)^{\frac{1}{2}}\)
  
  - **Kinematic boundary conditions:** \(\vec{v} = \vec{U}_b \to \vec{v}^* = \vec{U}_b^*\)
  
  - **Dynamic boundary conditions:**

  \[
p = p_o + \Delta p \quad \text{where} \quad \Delta p = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \sum \frac{1}{R_1} + \frac{1}{R_2} \frac{p_o-p_v}{(p_o-p_v) L} \sum \frac{\sum / \rho}{\sigma U^2 L}
  \]

- \(\frac{U^2 L}{\sum / \rho} = We = \text{Weber number} \sim \frac{\text{inertial forces}}{\text{surface tension forces}}\)

  note: \(L \gg R_o\) usually

Alternatively, using **physical arguments:** forces acting on a fluid particle

1. inertial forces \(\sim \text{mass} \times \text{acceleration} \sim (\rho L^3)\left(\frac{U^2}{L}\right) = \rho U^2 L^2\)
2. viscous forces \( \sim \frac{\mu}{\partial y} \times \text{area} \) \( (\mu \frac{U}{L} L^2) = \mu UL \)

3. gravitational forces \( \sim \) mass \( \times \) gravity \( \sim (\rho L^3)g \)

4. pressure forces \( \sim (p_o - p_v)L^2 \)

For similar streamlines, particles must be acted on by forces whose resultant is in the same direction at geosimilar points. Therefore, forces must be in the same ratios:

\[
\frac{\text{inertia}}{\text{viscous}} = \frac{\rho U^2 L^2}{\mu UL} = \frac{UL}{\nu} = Re
\]

\[
\left( \frac{\text{inertia}}{\text{gravity}} \right)^{1/2} \sim \left( \frac{\rho U^2 L^2}{\rho g L^3} \right)^{1/2} = \frac{U}{\sqrt{gL}} = Fr
\]

\[
\left( \frac{\text{1/2 inertia}}{\text{pressure}} \right)^{-1} \sim \left( \frac{p_o - p_v}{1/2 \rho U^2 L^2} \right) = \frac{p_o - p_v}{1/2 \rho U^2} = \sigma
\]

**Importance of Various Parameters**

- Govern flow similitude of different systems.
- Provide guidance and approximate the complex physical problem.

**e.g.**
Parameters:

\[
S = \frac{L}{U T}, \quad \sigma = \frac{P_o - P_v}{\frac{1}{2} \rho U^2}, \quad W_e = \frac{U^2 L}{\nu}, \quad F_r = \frac{U}{\sqrt{g L}}, \quad R_e = \frac{U L}{\nu}
\]

Force coefficient on the foil:

\[
C_F = \frac{F}{\frac{1}{2} \rho U^2 L^2} = C_F \left( S, \sigma^{-1}, W_e^{-1}, F_r, R_e^{-1} \right)
\]

1. \( S = L/UT \), change \( S \) with \( \sigma, W_e, F_r, R_e \) fixed.

For \( S < 1 \), assume steady-state: \( \frac{\partial}{\partial t} = 0 \)

For \( S >> 1 \), unsteady effect is dominant. For example:

\[
\begin{align*}
L & \approx 10 m \\
U & \approx 10 m/s \\
\Rightarrow T & \approx 1 \text{ sec} \quad \text{gives } S \approx 1, \therefore \text{ for } T >> 1 \text{sec assume steady state since } S << 1
\end{align*}
\]
So, for steady-state problem:

\[ C_F = C_F (\sigma^{-1}, W_e^{-1}, F_r, R_e^{-1}) \]

2. \( \sigma = \frac{P_o - P_v}{2\rho U^2} \) (fixed \( R_e, F_r \) and \( W_e \)).

\( P_v \): Vapor pressure
\( P_o \leq P_v \): State of fluid changes from liquid to gas ← CAVITATION

Mechanism: \( P_o < P_v \) → Fluids cannot withstand tensions, the state of fluids changes.

Consequence:
(1) Unsteady → Vibration of the structures, which may lead to fatigue
(2) Unstable → Sudden cavity collapses → huge force acting on the structure surface → surface erosion.

For \( \sigma \ll 1 \), there is cavitation, and for \( \sigma \gg 1 \), there is no cavitation. For example:
\[
\begin{align*}
\left\{ \begin{array}{l}
p_0 \approx 10^5 \text{N/m}^2 \\
p_v \approx 2 \times 10^3 \text{n/m}^2 \\
\rho \approx 10^3 \text{kg/m}^3 \\
L \approx 100 \text{m} \\
U \approx 10 \text{m/s}
\end{array} \right. \\
\Rightarrow \sigma = 2. \text{ To have cavitation we need large } U \text{ or } p_o \sim p_v \\
\end{align*}
\]

Note: \( p_v \) is the pressure at which the water boils.

For steady non-cavitation flow (\( \sigma \gg 1 \))

\[
C_F = C_F \left( W_e^{-1}, F_r, R_e^{-1} \right)
\]

3. \( W = \frac{U^2 L}{\rho} \) (fixed \( R_e \) and \( F_r \)). For example, if \( U = 1 \text{m/s} \), \( \sum = 0.07 \text{N/m (water-air 20}^\circ\text{C)} \), \( \rho = 10^3 \text{kg/m}^3 \) and \( L = 100 \text{ m} \), we end up with \( W_e \approx 10^8 \). If we want \( W_e \approx 1 \), we need \( L \approx 10^{-4} \text{ m} \). Then, for \( L \gg 10^{-4} \text{ m} \), \( W_e \to \infty \) and \( W_e^{-1} \to 0 \), so neglect surface tension effect.

For steady, non-cavitation, non-surface tension effect,

\[
C_F = C_F \left( F_r, R_e^{-1} \right)
\]

4. \( F_r = \frac{U}{\sqrt{g L}} \), which measures the effect of gravity.

For problems without dynamic boundary conditions (i.e. if free surface is absent) or if the free-surface is far away or not displaced, gravity effects are irrelevant and \( F_r \) is not important \( \to F^* = C_F \left( R_e^{-1} \right) \)

e.g.
In general $C_F = C_F (F_r, R_e^{-1}) = C_1 (F_r) + C_2 (R_e^{-1}) \quad \leftarrow$ Froude’s Hypothesis

Dynamic similarity requires:

$(R_e)_1 = (R_e)_2,$
$(F_r)_1 = (F_r)_2.$

For two geometrically similar systems $→ U_1 = U_2, L_1 = L_2$ for the same $\nu$ and $g$.

5. $R_e = UL/\nu.$

For steady, no $\sigma$, no $We$, no gravity effects, $C_F = C_F (R_e^{-1})$
\[ R_e \ll 1, \text{ Stokes flow (creeping flow)} \]
\[ R_e < (R_e)_{cr}, \text{ Laminar flow} \]
\[ R_e > (R_e)_{cr}, \text{ Turbulent flow} \]
\[ R_e \to \infty, \text{ Ideal flow} \]

For example:

\[
\begin{align*}
U &= 10 \text{m/s} \\
L &= 10 \text{m} \\
\nu &= 10^{-6} \text{m}^2/\text{sec}
\end{align*}
\]
\[ \Rightarrow R_e = 10^8 \text{ or } R_e^{-1} = 10^{-8} \]

For steady, no \( \sigma \), no \( W_e \), no gravity effect and ideal fluid:

\[ C_F = C_F (0,0,0,0) = \text{constant} = 0 \]

\( \rightarrow \text{D’Alembert’s Paradox: No drag force on moving body.} \)