Chapter 2 - Similitude

Similitude: Similarity of behavior of different systems.

Real world ↔ “model”
(prototype) (physical experiment, mathematical, computer, ...)

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For similitude, similarity parameters (SP’s) required to be the same for the model and the real world

2.1 - Dimensional Analysis (DA) to Obtain Similarity Parameters (SP’s)

Buckingham’s $\pi$ theory:

Reduce number of variables → derive dimensionally homogeneous relationships.

1. Specify (all) the (say N) relevant variables (dependent or independent): $x_1, x_2, \ldots x_N$
e.g. time, force, fluid density, distance...
We want to relate the $x_i$’s to each other $I(x_1, x_2, \ldots x_N) = 0$
2. Identify (all) the (say P) relevant basic physical units (“dimensions”) e.g. M,L,T (P = 3) [temperature, charge, ...].

3. Let \( \pi = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N} \) be a dimensionless quantity formed from the \( x_i \)'s. Suppose \( x_1 = C_i M^{m_i} L^{l_i} T^{t_i}, i = 1, 2, \ldots, N \) where the \( C_i \) are dimensionless constants. For example, if \( x_1 = KE = \frac{1}{2} M V^2 = \frac{1}{2} M^1 L^2 T^{-2} \) (kinetic energy), we have that \( C_1 = \frac{1}{2}, m_1 = 1, l_1 = 2, t_1 = -2 \). Then

\[
\pi = (C_1^{\alpha_1} C_2^{\alpha_2} \ldots C_N^{\alpha_N}) M^{\alpha_1 m_1 + \alpha_2 m_2 + \ldots + \alpha_N m_N} L^{\alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_N l_N} T^{\alpha_1 t_1 + \alpha_2 t_2 + \ldots + \alpha_N t_N}
\]

For \( \pi \) to be dimensionless, we require

\[
\begin{align*}
\sum_{i=1}^{P} \alpha_i m_i &= 0 \\
\sum_{i=1}^{P} \alpha_i l_i &= 0 \\
\sum_{i=1}^{P} \alpha_i t_i &= 0
\end{align*}
\]

\[\text{aP} \times N \text{ system of Linear Equations (1)}\]

Since (1) is homogeneous, it always has a trivial solution,

\[
\alpha_i \equiv 0, i = 1, 2, \ldots, N \text{ (i.e.} \pi \text{ is constant)}
\]

There are 2 possibilities:

(a) (1) has no nontrivial solution (only solution is \( \pi = \text{constant}, \text{i.e. independent of} x_i \)'s), which implies that the \( N \) variable \( x_i, i = 1, 2, \ldots, N \) are Dimensionally Independent (DI), i.e. they are “unrelated” and “irrelevant” to the problem.

(b) (1) has \( J (J > 0) \) nontrivial solutions, \( \pi_1, \pi_2, \ldots, \pi_J \). In general, \( J < N \), in fact, \( J = N - K \) where \( K \) is the rank or “dimension” of the system of equations (1).
Model Law:
Instead of relating the \( N \) \( x_i \)'s by \( \mathcal{I}(x_1, x_2, \ldots x_N) = 0 \), relate the \( J \) \( \pi \)'s by
\[
F(\pi_1, \pi_2, \ldots \pi_J) = 0, \text{ where } J = N - K < N
\]
For similitude, we require
\[
(\pi_{\text{model}})_j = (\pi_{\text{prototype}})_j \text{ where } j = 1, 2, \ldots, J.
\]
If 2 problems have all the same \( \pi_j \)'s, they have similitude (in the \( \pi_j \) senses), so \( \pi \)'s serve as similarity parameters.

Note:
- If \( \pi \) is dimensionless, so is constant \( \times \pi \), \( \pi^{\text{const}} \), \( 1/\pi \), etc.
- If \( \pi_1, \pi_2 \) are dimensionless, so is \( \pi_1 \times \pi_2 \), \( \frac{\pi_1}{\pi_2}, \pi_1^{\text{const1}} \times \pi_2^{\text{const2}}, \text{ etc.} \)

In general, we want the set (not unique) of independent \( \pi_j \)'s, for e.g., \( \pi_1, \pi_2, \pi_3 \) or \( \pi_1 \times \pi_2, \pi_3 \), but not \( \pi_1, \pi_2, \pi_1 \times \pi_2 \).

Example:
Application of Buckingham \( \pi \) Theory.

![Diagram](https://via.placeholder.com/150)

Figure 1: Force on a smooth circular cylinder in steady incompressible fluid (no gravity)
\[ x_i : F, U, D, \rho, \nu \rightarrow N = 5 \]
\[ x_i = c_i M^{m_i} L^{l_i} T^{t_i} \rightarrow P = 3 \]

\[
\begin{array}{c|ccccc}
& F & U & D & \rho & \nu \\
\hline
P = 3 & m_i & 1 & 0 & 0 & 1 & 0 \\
& l_i & 1 & 1 & 1 & -3 & 2 \\
& t_i & -2 & -1 & 0 & 0 & -1 \\
\end{array}
\]

\[ \pi = \Gamma^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5} \]

For \( \pi \) to be non-dimensional, the set of equations

\[
\begin{align*}
\alpha_i m_i &= 0 \\
\alpha_i l_i &= 0 \\
\alpha_i t_i &= 0 
\end{align*}
\]

has to be satisfied. The system of equations above after we substitute the values for the \( m_i \)'s, \( l_i \)'s and \( t_i \)'s assume the form:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & -3 & 2 \\
-2 & -1 & 0 & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\]

The rank of this system is \( K = 3 \), so we have \( j = 2 \) nontrivial solutions. Two families of solutions for \( \alpha_i \) for each fixed pair of \((\alpha_4, \alpha_5)\), exists a unique solution for \((\alpha_1, \alpha_2, \alpha_3)\). We consider the pairs \((\alpha_4 = 1, \alpha_5 = 0)\) and \((\alpha_4 = 0, \alpha_5 = 1)\), all other cases are linear combinations of these two.
1. Pair $\alpha_4 = 1$ and $\alpha_5 = 0$.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} =
\begin{pmatrix}
-1 \\
4 \\
2
\end{pmatrix}
\]

which has solution

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} =
\begin{pmatrix}
-1 \\
2 \\
2
\end{pmatrix}
\]

\[\therefore \pi_1 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} u^{\alpha_5} = \frac{\rho U^2 D^2}{F} \]

Conventionally, $\pi_1 \to 2\pi_1^{-1}$ and $\therefore \pi_1 = \frac{F}{2\rho U^2 D^2} \equiv C_d$, which is the Drag coefficient.

2. Pair $\alpha_4 = 0$ and $\alpha_5 = 1$.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
-2 \\
-1
\end{pmatrix}
\]

which has solution

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
-1 \\
-1
\end{pmatrix}
\]

\[\therefore \pi_2 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} u^{\alpha_5} = \frac{v}{U D} \]

Conventionally, $\pi_2 \to \pi_2^{-1}$, $\therefore \pi_2 = \frac{U D}{v} \equiv R_e$, which is the Reynolds number.

Therefore,

\[
F(\pi_1, \pi_2) = 0 \quad \text{or} \quad \pi_1 = f(\pi_2)
\]
\[
F(C_d, R_e) = 0 \quad \text{or} \quad C_d = f(R_e)
\]
\[
F\left(\frac{F}{2\rho U^2 D^2}, \frac{U D}{v}\right) = 0 \quad \text{or} \quad \frac{F}{2\rho U^2 D^2} = f\left(\frac{U D}{v}\right)
\]