### Introduction

Governing Equations so far:

<table>
<thead>
<tr>
<th>Knowns</th>
<th>Number of Equations</th>
<th>Number of Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>Continuity (conservation of mass)</td>
<td>1</td>
</tr>
<tr>
<td>$F_i$</td>
<td>Euler (conservation of momentum)</td>
<td>3</td>
</tr>
</tbody>
</table>

Therefore, some constitutive relationships are needed to relate $v_i$ to $\tau_{ij}$.

### 1.7 Newtonian Fluid

1. Consider a fluid at rest ($v_i \equiv 0$). Then according to Pascal’s Law:

   $$\tau_{ij} = -p_s \delta_{ij} \text{ (Pascal’s Law)}$$

   $$\tau = \begin{bmatrix} -p_s & 0 & 0 \\ 0 & -p_s & 0 \\ 0 & 0 & -p_s \end{bmatrix}$$

   where $p_s$ is the hydrostatic pressure and $\delta_{ij}$ is the Kroenecker delta function, equal to 1 if $i = j$ and 0 if $i \neq j$.

2. Consider a fluid in motion. The fluid stress is defined as:

   $$\tau_{ij} = -p \delta_{ij} + \hat{\tau}_{ij}$$
where \( p \) is the thermodynamic pressure and \( \hat{\tau}_{ij} \) is the dynamic stress, which is related to the velocities empirically.

Experiments with a wide class of fluids, "Newtonian" fluids, obtain that:

\[
\hat{\tau}_{ij} \approx \text{linear function of the rate of strain} \equiv \text{velocity gradient}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial X}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial X}{\partial t} \right)
\]

i.e. \( \hat{\tau}_{ij} \approx \sum_{k=1}^{3} \mathbf{\alpha}_{ijkm} \frac{\partial u_k}{\partial x_m} \), \( i, j, k, m = 1, 2, 3 \)

\( \mathbf{\alpha}_{ijkm} \) are empirical coefficients (constants for Newtonian fluids)

Note that the shear stress is proportional to the rate of strain.

\[
\begin{array}{c}
\text{Newtonian Fluid} \\
\text{Fluid}
\end{array}
\]

For isotropic fluids, this reduces to:

\[
\hat{\tau}_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \left( \nabla \cdot \vec{v} \right)
\]

where the fluid properties are:

- \( \mu \) - (coefficient of) dynamic viscosity.
- \( \lambda \) - bulk elasticity, ‘second’ coefficient of viscosity
For incompressible flow, \( \frac{\partial u_i}{\partial x_i} = 0 \). Therefore, for an incompressible, isotropic, Newtonian fluid the viscous stress \( \hat{\tau}_{ij} \) is given as

\[
\hat{\tau}_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

### 1.8 Navier-Stokes equations

<table>
<thead>
<tr>
<th>Equations</th>
<th>Number of Equations</th>
<th>Unknowns Number of Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>continuity</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>constitutive</td>
<td>6 (symmetry)</td>
<td>6</td>
</tr>
<tr>
<td>(Newtonian)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

|               |                     |                             |
|               | 10                  | 10                          |

Substitute the equation for the stress tensor

\[
\tau_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

for a Newtonian fluid into Euler’s equation:

\[
\rho \frac{Du_i}{Dt} = F_i + \frac{\partial \tau_{ij}}{\partial x_j}
\]

where

\[
\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \left( \frac{\partial \frac{\partial u_i}{\partial x_j}}{\partial x_j} + \frac{\partial \frac{\partial u_j}{\partial x_i}}{\partial x_i} \right) + \frac{\partial^2 u_j}{\partial x_j \partial x_j} + \frac{\partial \partial u_i}{\partial x_j \partial x_i} \frac{\partial u_j}{\partial x_j}
\]

and \( \frac{\partial u_i}{\partial x_j} = 0 \) due to continuity. Finally,
\[
\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\rho} F_i \quad \text{Tensor form}
\]
\[
\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \frac{1}{\rho} \vec{F} \quad \text{Vector form}
\]

where \( \nu \equiv \frac{\mu}{\rho} \) denoted as the **Kinematic viscosity** \( [L^2/T] \).

- Navier-Stokes equations for incompressible, Newtonian fluids

<table>
<thead>
<tr>
<th>Equation</th>
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</tr>
</thead>
<tbody>
<tr>
<td>continuity</td>
<td>1</td>
<td>( p )</td>
</tr>
<tr>
<td>Navier-Stokes</td>
<td>3 (symmetry)</td>
<td>( u_i )</td>
</tr>
</tbody>
</table>

| 1.9 Boundary Conditions |

1. **Kinematic Boundary Conditions**: Specifies kinematics (position, velocity, …) On a solid boundary, velocity of the fluid = velocity of the body. i.e. velocity continuity.

\[ \vec{v} = \vec{u} \] "no-slip" boundary condition

where \( \vec{v} \) is the fluid velocity at the body and \( \vec{u} \) is the body surface velocity

- \( \vec{v} \cdot \hat{n} = \vec{u} \cdot \hat{n} \) no flux – continuous flow
- \( \vec{v} \cdot \hat{t} = \vec{u} \cdot \hat{t} \) no slip – finite shear stress
2. **Dynamic Boundary Conditions**: Specifies dynamics (pressure, shear stress, ...)

Stress continuity:

\[
p = p' + p_{\text{interface}}
\]
\[
\tau_{ij} = \tau'_{ij} + \tau_{ij \text{ interface}}
\]

The most common example of interfacial stress is surface tension.

---

**Surface Tension**

- Notation: \( \Sigma \) [Tension force / Length] \( \equiv \) [Surface energy / Area].
- Surface tension is due to the inter molecular forces attraction forces in the fluid.
- At the interface of two fluids, surface tension implies in a pressure jump across the interface. \( \Sigma \) gives rise to \( \Delta p \) across an interface.
- For a water/air interface: \( \Sigma = 0.07 \text{ N/m} \). This is a function of temperature, impurities etc...
- 2D Example:
  \[
  \cos \frac{d\theta}{2} \cdot \Delta p \cdot Rd\theta = 2\Sigma \sin \frac{d\theta}{2} \approx 2\Sigma \frac{d\theta}{2} \approx \frac{d\theta}{2}
  \]
  \[
  \therefore \Delta p = \frac{\Sigma}{R}
  \]

Higher curvature implies in higher pressure jump at the interface.
• 3D Example: Compound curvature

\[ \Delta p = \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \Sigma \]

where \( R_1 \) and \( R_2 \) are the principle radii of curvature.

### 1.10 Body Forces – Gravity

• Conservative force:

\[ \vec{F} = -\nabla \varphi \quad \text{for some } \varphi, \]

where \( \varphi \) is the force potential.

\[ \oint \vec{F} \cdot d\vec{x} = 0 \quad \text{or} \quad \int_{1}^{2} \vec{F} \cdot d\vec{x} = -\int_{1}^{2} \nabla \varphi \cdot d\vec{x} = \varphi(\vec{x}_1) - \varphi(\vec{x}_2) \]

• A special case of a conservative force is gravity.

\[ \vec{F} = -\rho g \hat{k}, \]

with gravitational potential:

\[ \varphi = \rho g z \rightarrow \vec{F} = -\nabla \varphi = \nabla(-\rho g z) = -\rho g \hat{k}, \]

where \(-\rho g z\) is the hydrostatic pressure \( p_s = -\rho g z = -\varphi \).
Navier-Stokes equation:

\[
\rho \frac{D\vec{v}}{Dt} = -\nabla p + \vec{F} + \rho \nu \nabla^2 \vec{v}
\]

\[
= -\nabla p - \rho g z + \rho \nu \nabla^2 \vec{v}
\]

\[
= -\nabla (p + \rho g z) + \rho \nu \nabla^2 \vec{v},
\]

but \(p - p_s = p_d\) and \(p_s = -\rho g z\), where \(p\) is the total pressure and \(p_d\) is the dynamic pressure. Therefore,

\[
\rho \frac{D\vec{v}}{Dt} = -\nabla p_d + \rho \nu \nabla^2 \vec{v}
\]

- Presence of gravity body force is equivalent to replacing total pressure \(p\) by dynamic pressure \(p_d\) in the Navier-Stokes (N-S) equation.

- Solve the N-S equation with \(p_d\), then calculate \(p = p_d + p_s = p_d - \rho g z\).