Vortex Shedding and Vortex Induced Vibrations

Consider a steady flow $U_o$ on a bluff body with diameter $D$.

We would expect the average forces to be:

The measured oscillatory forces are:
**Von Karman Street**: Unsteady non-symmetric wake of staggered array of vortices.

Frequency of vortex shedding $f = \omega/2\pi$ is given by a non-dimensional number.

$$\frac{fD}{U_0} = S(Re)$$

where $f$ is the Strouhal frequency, $D$ is the body diameter and $S$ is the Strouhal number. The Drag $F_x$ has frequency $2f$ and non-zero mean value, and the Lift $F_y$ has frequency $f$, but zero mean value. For laminar flow $S \sim 0.22$ for a cylinder, and for turbulent flow, $S \sim 0.3$ for a cylinder.
\( C_D \) and \( C_L \) are functions of the correlation length. For "\( \infty \)" correlation length, \( C_L \sim O(1) \) for a fixed cylinder, comparable to \( C_D \). For a moving cylinder, if the Strouhal frequency \( f \) is close to one of the cylinder natural frequencies, lock-in occurs. Therefore, if one natural frequency is close to the Strouhal Frequency \( f_S \), we have large amplitude of motion \( \Rightarrow \) \textbf{Vortex induced vibration} (VIV).

\textbf{4.2 – Drag on a very streamlined body: Flat Plate}

\[
\frac{D}{\frac{1}{2} \rho U^2 (Lb)} = C_f (Re, L/b)
\]

Unlike a bluff body, \( C_f \) is a strong function of \( Re \) since \( D \) is proportional to \( \nu \). \( \tau = \nu \frac{\partial u}{\partial y} \)

\textbf{Flat Plate Drag}

- \( Re \) depends on plate smoothness, ambient turbulence, …
In general, $C_f$’s are much smaller than $C_D$’s (a factor of 10 : 100). Therefore, designing streamlined bodies allows minimal separation and form drag (at the expense of friction drag).

In general, for streamlined bodies

$$C_{force} \text{ is a combination of } C_D(Re) \text{ and } C_f(Re)$$

where $C_D$ is a function of the regime and $C_f$ is a function of $Re_L$ continuously.

**Governing equations:**

- Navier-Stokes’:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \frac{1}{\rho} \vec{f}$$

- Conservation of mass:

$$\nabla \cdot \vec{v} = 0$$

- Boundary conditions on solid boundaries “no-slip”:

$$\vec{v} = \vec{U}$$

Equations very difficult to solve, analytic solution only for a few very special cases (usually when $(\vec{v} \cdot \nabla) \vec{v} = 0$...)

**4.3 Steady Laminar Flow Between 2 Infinite Parallel Walls - Plane Couette Flow**
Assume steady flow \( \frac{\partial}{\partial t} = 0 \). For the horizontal dimensions \((x, z) >> h\), we assume flow independent of \(x\) and \(z\), i.e., \(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial z} = 0\), so \(\vec{v} = \vec{v}(y)\).

- Kinematic boundary conditions (k.b.c.):
  \[
  \vec{v} = (0, 0, 0) \text{ on } y = 0 \\
  \vec{v} = (U, 0, 0) \text{ on } y = h
  \]

- Conservation of mass:
  \[
  \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow \frac{\partial v}{\partial y} = 0 \rightarrow v = v(x, z),
  \]
  but \(v = 0\) on \(y = 0, h\) (k.b.c.), therefore \(v = 0\).

- Navier-Stokes equation for steady flow \( \frac{\partial}{\partial t} = 0 \), no \(f\) and \(\frac{\partial v}{\partial x} = \frac{\partial v}{\partial z} = 0\):
  \[
  u : \nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho} \frac{\partial p}{\partial x} \\
  v : \frac{\partial p}{\partial y} = 0 \rightarrow p = p(x, z) \\
  w : \nu \frac{\partial^2 w}{\partial y^2} = \frac{1}{\rho} \frac{\partial p}{\partial z}
  \]
  We assume that \(p = p(x)\), i.e. \(\frac{\partial p}{\partial z} \equiv 0\), then \(\nu \frac{\partial^2 w}{\partial y^2} = 0 \rightarrow w = a + by\). But k.b.c.: \(w = 0\) on \(y = 0, h\). Therefore, \(w \equiv 0\).

Finally: \(v = w = 0, u = u(y), p = p(x)\)

\[
\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx} \text{ where } \mu = \rho \nu
\]

- Solution:
  \[
  u = \frac{1}{2} y^2 \frac{1}{\mu} \frac{dp}{dx} + C_1 + C_2 y
  \]
k.b.c.: $C_1 = 0$ and $C_2 = \left( U - \frac{1}{2}h^2 \frac{dp}{dx} \right)$ since $u(0) = 0$ and $u(h) = U$. Finally,

$$u = \frac{1}{2\mu} \left( y - h \right) y \frac{dp}{dx} + \frac{U y}{h} \quad \text{(plane) Couette flow}$$

### 4.4 - Steady Laminar Flow in a pipe - Poiseuille Flow.

Assume steady, and for $L >> a$, $\frac{\partial \vec{v}}{\partial x} = \frac{\partial \vec{v}}{\partial \theta} \equiv 0 \rightarrow \vec{v} = \vec{v}(r)$, $r^2 = y^2 + z^2$.

Can show:

$$\vec{v} = (v_x, v_r, v_\theta)$$

$$v_r = v_\theta = 0, \quad v_x = v_x(r), \quad p = p(x)$$

$$\frac{1}{\rho} \frac{dp}{dx} = \nu \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{dv_x}{dr} \right) \right)$$

$r$ component of $\nabla^2$ in cylindrical coordinates

K.b.c.: $v_x(a) = 0$ (no slip) and $\frac{dv_x}{dr}(0) = 0$ (symmetry).
Solution:

\[ v_x(r) = \frac{1}{4\mu} \left( -\frac{dp}{dx} \right) (a^2 - r^2) \] Poiseuille flow

4.5 Unsteady Flow (boundary layer growth) over an infinite flat plate

For steady \((\text{Couette Poiseuille})\) flow, vorticity, viscosity effects diffuse to all \((h/a)\)

1. limit \(x\)

2. limit \(t\) (§4.5)

Consider the simplest example of an infinite plate in unsteady motion:

Assuming \(\nabla p = 0\), we have \(\nabla \frac{\partial v}{\partial z}, \frac{\partial v}{\partial x} = 0\), so \(\vec{v} = \vec{v}(y, t)\)

Can show that \(v = w = 0\) and \(u = u(y, t)\).
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0 \]

Finally:

\[ \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{(velocity diffusion equation)} \]  

(1)

B.c.: \( u(0, t) = U(t), t > 0; u \) bounded \((\to 0)\) as \( y \to \infty \) + suitable initial condition.

4.5.1 Sinusoidally Oscillating Plate

\( U(t) = U_o \cos \omega t = \text{Real} \{ U_o e^{i \omega t} \} \quad e^{i \alpha} = \cos \alpha + i \sin \alpha \) where \( \alpha \) is real. Let \( u(y, t) = \text{Real} \{ f(y) e^{i \omega t} \} \) where \( f(y) \) is an unknown complex (magnitude & phase) amplitude. Then (1):

\[ i \omega f = \nu \frac{d^2 f}{dy^2} \quad \text{2nd order ODE for} f(y) \]

General Solution:

\[ f(y) = C_1 e^{(1+i)\left(\sqrt{\omega/2\nu}\right)y} + C_2 e^{-(1+i)\left(\sqrt{\omega/2\nu}\right)y} \]

B.c.: \( u \to \) bounded as \( y \to \infty \), \( C_1 = 0 \). \( u \to U(t) \) as \( y = 0 \), \( C_2 = U_o \).

Finally:

\[ u(y, t) = U_o e^{-\left(\sqrt{\omega/2\nu}\right)y} \cos \left( -\sqrt{\frac{\omega}{2\nu}} y + \omega t \right) \quad \text{Stokes' (Oscillatory) b.l.} \]
4.5.2 Impulsively Started Plate

\[ u(y,t) : \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \]

B.c.:

\[ \begin{align*}
    u(o,t) &= U_o \\
u(\infty,t) &= 0 \end{align*} \]

for \( t > 0 \), i.e. \( u(y,0) = 0 \)

Problem has no explicit time scale, can use dimensional analysis to solve in terms of a similarity parameter:

\[ \frac{u}{U_o} = f(y,t,\nu) = f\left(\frac{y}{2\sqrt{\nu t}}\right) ; \text{i.e.} \quad \frac{u}{U_o} = f(\eta) \]

\( \equiv \eta \) similarity parameter

Self similar solution

Solution:

\[ \frac{u}{U_o} = erfc(\eta) = 1 - erf(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-x^2} \, dx \]

Impulsively started flat-plate boundary layer solution

Complementary error function