Some Properties of Added-Mass Coefficients

1. \( m_{ij} = \rho \text{[function of geometry only]} \)

\[ F, M = [\text{linear function of } m_{ij}] \times [\text{function of instantaneous } U, \dot{U}, \Omega] \]

2. Relationship to momentum of fluid:

Where we use: \( \Phi \) for \( U = 1 \) and \( \phi = U\Phi \).

Linear momentum \( \vec{L} \) in fluid:

\[
\vec{L} = \iiint_V \rho \vec{v} dV = \iiint_V \rho \nabla \phi dV = \iiint_B \rho \Phi \vec{n} dS
\]

\[
L_x(t = T) = \iint_B \rho \Phi n_x dS
\]

Force on fluid by body = \(-F(t) = -(m_A \dot{U}) = m_A \dot{U} \).
\[
\int_{0}^{T} dt \left[ -F(t) \right] = \int_{0}^{T} m_A \dot{U} dt = m_A U \left[ \frac{1}{m_A} \right]_0^T = \int_{B} \rho \Phi n_x dS
\]

Therefore, \( m_A = \text{total fluid momentum for body moving at } U = 1 \) (regardless of how we get there from rest) = fluid momentum per unit velocity of body.

\[
\frac{\partial \phi}{\partial n} = \nabla \Phi \cdot \hat{n} = (u, 0, 0) \cdot \hat{n} = U n_x, \quad \frac{\partial \Phi}{\partial n} = n_x \text{ for } U = 1.
\]

\[
m_A = \rho \int_{B} \Phi \frac{\partial \Phi}{\partial n} dS
\]

For general 6 DOF:

\[
m_{ji} = \rho \int_{B} \Phi_i \frac{\partial \Phi_j}{\partial n} dS = \rho \int_{B} \Phi_i \frac{\partial \Phi_j}{\partial n} dS = j \text{ fluid momentum due to } i \text{ body motion}
\]

3. Symmetry of added mass matrix \( m_{ij} = m_{ji} \).

\[
m_{ji} = \rho \int_{B} \Phi_i \left( \frac{\partial \Phi_j}{\partial n} \right) dS = \rho \int_{B} \Phi_i (\nabla \Phi_j \cdot \hat{n}) dS = \rho \int_{V} \nabla \cdot (\Phi_i \nabla \Phi_j) dV
\]

\[
= \rho \int_{V} \left( \nabla \Phi_i \cdot \nabla \Phi_j + \Phi_i \nabla^2 \Phi_j dV \right)
\]

Therefore,

\[
m_{ji} = \rho \int_{V} \nabla \Phi_i \cdot \nabla \Phi_j dV = m_{ij}
\]
4. Relationship to kinetic energy of fluid. In general, for body motion $U_i\{(U_1, U_2, \ldots, U_6)\}$.

$$\phi = \sum_{ notation }^{} U_i \Phi_i ; \Phi_i = \text{potential for } U_i = 1$$

$$K.E. = \frac{1}{2} \rho \int \int \int \nabla \phi \cdot \nabla \phi dv = \frac{1}{2} \rho \int \int \int U_i \nabla \Phi_i \cdot U_j \nabla \Phi_j dv$$

$$= \frac{1}{2} \rho U_i U_j \int \int \int \nabla \Phi_i \cdot \nabla \Phi_j dv = \frac{1}{2} m_{ij} U_i U_j$$

K.E. depends only on $m_{ij}$ and instantaneous $U_i$.

5. Use of symmetry to simplify $m_{ij}$. From 36 $\rightarrow$ 21 $\rightarrow$ ? Choose coordinate system so that some $m_{ij} = 0$ by symmetry.

e.g. example 1: port-starboard symmetry (sym w.r.t. $x_3$)

$$m_{ij} =$$

\[
\begin{bmatrix}
    m_{11} & m_{12} & 0 & 0 & 0 & m_{16} \\
    m_{22} & 0 & 0 & 0 & m_{26} \\
    m_{33} & m_{34} & m_{35} & 0 & 0 \\
    m_{44} & 0 & m_{45} & 0 & 0 \\
    m_{55} & 0 & 0 & m_{56} & 0 \\
    U_1 & U_2 & U_3 & \Omega_1 & \Omega_2 \\
\end{bmatrix}
\begin{bmatrix}
    F_x \\
    F_y \\
    F_z \\
    M_x \\
    M_y \\
    M_z \\
\end{bmatrix}
\]

12 independent coefficients

\[ m_{13} = 0 \text{ by symmetry} \therefore \text{ NOT OBVIOUS!} \]
example 2: rotational (axi) sym. about $x_1$

\[
\begin{bmatrix}
  m_{11} & 0 & 0 & 0 & 0 & 0 \\
  m_{22} & 0 & 0 & 0 & m_{35} & \\
  m_{22} & 0 & m_{35} & 0 & \\
  0 & 0 & 0 & \\
  m_{55} & 0 & \\
  m_{55} &
\end{bmatrix}
\]

where $m_{22} = m_{33}, m_{55} = m_{66}$ and $m_{26} = m_{35}$, so 4 different coefficients

**Exercise:** How about 3 planes of symmetry (e.g. a cuboid); a cube; a sphere?? Work out detail!

### 3.12 - Slender body Approximation.

Estimating $m_{ij}$ of a slender 3D body using 2D strip-wise $M_{ij}$. 
Idea “\( m_{ij} = \text{sum } [M_{ij}(x) \text{ contributions}] \)”

e.g.

\[
m_{33} = \int_L M_{33}(x) \, dx; m_{22}, m_{33}, \text{ etc.}
\]

Yaw moment due to sway acceleration:

\[
m_{53} = \int_L (-x) M_{33}(x) \, dx
\]

In general: Moment_5 = \((-x)\) force_3(x) and \(\dot{U}_3(x) = (-x) \dot{\Omega}_5\)

\[
m_{55} = \int_L (-x)(-x) M_{33}(x) \, dx = \int_L x^2 M_{33}(x) \, dx
\]

Similarly for \(m_{22}, m_{44}, m_{42}, \ldots\) How about \(m_{23}, m_{25}\) ?? (Work out the detail!!)

**Buoyancy Effects Due to Accelerating Flow**

**Example 2**: Force on a stationary sphere in a fluid that is accelerated against it.
\[ \phi (r, \theta, t) = U(t) \left( r + \frac{a^3}{2r^2} \right) \cos \theta \]

\[ \frac{\partial \phi}{\partial t} \bigg|_{r=a} = \dot{U} \frac{3a}{2} \cos \theta \]

\[ \nabla \phi \big|_{r=a} = \left( 0, -\frac{3}{2} U \sin \theta, 0 \right) \]

\[ \frac{1}{2} |\nabla \phi|^2 \big|_{r=a} = \frac{9}{8} U^2 \sin^2 \theta \]

Then,

\[ F_x = (-\rho) \left( 2\pi r^2 \right) \int_0^\pi d\theta \sin \theta (-\cos \theta) \left[ \dot{U} \frac{3a}{2} \cos \theta + \frac{9}{8} U^2 \sin^2 \theta \right] \]

\[ = \dot{U} 3\pi \rho a^3 \int_0^\pi d\theta \sin \theta \cos^2 \theta + \rho U^2 \frac{9\pi}{4} a^2 \int_0^\pi d\theta \cos \theta \sin^3 \theta \]

\[ F_x = \dot{U} \rho \left( 2\pi a^3 \right) \]

\[ \frac{4\pi a^3 \rho + 2\pi a^3 \rho}{4\pi a^3 \rho + 2\pi a^3 \rho} = \rho \nu \]

Example (1)

<table>
<thead>
<tr>
<th>( U(t) )</th>
<th>(2)</th>
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<tbody>
<tr>
<td>![image]</td>
<td>![image]</td>
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</table>

\[ m_{(1)} < m_{(2)} \]

\[ m_{(1)} + \rho \forall \]
Part of $F_x$ is due to the pressure gradient which must be present to cause the fluid to accelerate:

$$U(t): \text{N.S. } \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} + w \frac{\partial U}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \text{ (ignore gravity)}$$

$$\frac{dp}{dx} = - \rho \dot{U} \text{ for uniform (1D) accelerated flow}$$

Force on the body due to the pressure field

$$\vec{F} = \int \int_B \hat{p} \hat{n} dS = - \int \int \int_{V_B} \nabla p dV; F_x = - \int \int \int_{V_B} \frac{\partial p}{\partial x} dV = \rho \nu \dot{U}$$

“Buoyancy” force due to pressure gradient $= \rho \nu \dot{U}$

Analogue: Buoyancy force due to hydrostatic pressure gradient. $g = \text{gravitational acceleration} \leftrightarrow \dot{U} = \text{fluid acceleration}$.

$$p_s = - \rho g y$$
$$\nabla p_s = - \rho g \hat{j} \rightarrow \vec{F}_s = - \rho g \nu \dot{\hat{j}} \text{ Archimedes principle}$$

Summary: Total force on a fixed sphere in an accelerated flow

$$F_x = \dot{U} \left( \begin{array}{c} \rho \nu \\ \text{Buoyancy} \\ m_{(1)} \\ \text{added mass} \\ \frac{1}{2} \rho \nu \end{array} \right) = \dot{U} \frac{3}{2} \rho \nu = \dot{U} 3m_{(1)}$$
In general, for any body in an accelerated flow:

\[ F_x = F_{\text{buoyancy}} + \dot{U}m_{(1)}, \]

where \( m_{(1)} \) is the added mass in still water (from now on, \( m \))

\[ F_x = -\dot{U}m \] for body acceleration with \( \dot{U} \)

**Added mass coefficient**

\[ c_m = \frac{m}{\rho d} \]

in the presence of accelerated flow \( C_m = 1 + c_m \)