3.11 Unsteady Motion - Added Mass

D’Alembert: ideal, irrotational, unbounded, steady.

Example 1: Force on a sphere accelerating \(U = U(t)\), unsteady) in an unbounded fluid at rest. (at infinity)

\[
\phi = -U(t) \frac{a^3}{2r^2} \cos \theta
\]

K.B.C on sphere: \(\frac{\partial \phi}{\partial r}\bigg|_{r=a} = U(t) \cos \theta\)

Solution: Simply a 3D dipole (no stream)

\[
\phi = -U(t) \frac{a^3}{2r^2} \cos \theta
\]

Check: \(\frac{\partial \phi}{\partial r}\bigg|_{r=a} = U(t) \cos \theta\)

Hydrodynamic force:
\[ F_x = -\rho \int_B \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS \]

On \( r = a \):

\[ \frac{\partial \phi}{\partial t} \bigg|_{r=a} = -\dot{U} \frac{a^3}{2r^2} \cos \theta \bigg|_{r=a} = -\frac{1}{2} \dot{U} a \cos \theta \]

\[ \nabla \phi \bigg|_{r=a} = \left( U \cos \theta, \frac{1}{2} U \sin \theta, 0 \right) \]

\[ V_r = \frac{\partial \phi}{\partial r}, \quad V_{\theta} = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad V_{\phi} = \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \phi} \]

\[ |\nabla \phi|^2 \bigg|_{r=a} = U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta; \quad \hat{n} = -\hat{e}_r, n_x = -\cos \theta \]

\[ \int_B \int dS = \int_0^\pi \left( a d\theta \right) \left( 2\pi a \sin \theta \right) \]

Finally,
\[ F_x = (-\rho) 2\pi a^2 \int_0^\pi d\theta \left( \sin \theta \right) \left( \cos \theta \right) \left( -\frac{\dot{U}a \cos \theta + \frac{1}{2} \left( U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta \right) \left| \nabla \phi \right|^2}{\pi a} \right) \]

\[ = -\pi \rho \dot{U} a^3 \int_0^{2/3} d\theta \sin \theta \cos^2 \theta + \rho U^2 \pi a^2 \int_0^\pi d\theta \sin \theta \cos \theta \left( \cos^2 \theta + \frac{1}{4} \sin^2 \theta \right) \]

\[ F_x = -\dot{U}(t) \left[ \frac{2}{3} \frac{\pi a^3}{\text{units of mass}} \right] \quad F_x = 0 \text{ if } \dot{U} = 0 \text{ steady (D’Alembert’s Condition)} \]

General 6 degrees of freedom motions

Added mass matrix (tensor)

\[ m_{ij}; i, j = 1, 2, 3, 4, 5, 6 \]

\[ m_{ij}: \text{associated with force on body in } i \text{ direction due to unit acceleration in } j \text{ direction. For example, for a sphere:} \]

\[ m_{11} = m_{22} = m_{33} = \frac{1}{2} \rho a^3 \quad \text{(all other } m_{ij} = 0) \]

Some added masses of simple 2D geometries

- circle

\[ m_{11} = m_{22} = \rho a^2 = \rho \pi a^2 \]
- ellipse

\[ m_{11} = \rho \pi a^2, \quad m_{22} = \rho \pi b^2 \]

- plate

\[ m_{11} = \rho \pi a^2, \quad m_{22} = 0 \]

- square

\[ m_{11} = m_{22} \approx 4.754\rho a^2 \]
A reasonable estimate for added mass of a 2D body is to use the displaced mass \((\rho \bar{v})\) of an “equivalent cylinder” of the same lateral dimension or one that “rounds off” the body. For example, we consider a square:

1. inscribed circle: \(m_A = \rho \pi a^2 = 3.14\rho a^2\).

\[
\begin{align*}
\text{circle inscribed in a square} & \\
2a & \quad a
\end{align*}
\]

2. circumscribed circle: \(m_A = \rho \rho \pi (\sqrt{2}a)^2 = 6.28\rho a^2\).

\[
\begin{align*}
\text{circle circumscribed about a square} & \\
(\sqrt{2})a & \quad a
\end{align*}
\]

Arithmetic mean of 1) + 2) \(\approx 4.71\rho a^2\).
General 6 degrees of freedom forces and moments on a rigid body moving
in an unbounded fluid (at rest at infinity)

\[ \vec{U}(t) = (U_1, U_2, U_3) \text{ Translation velocity} \]
\[ \vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6) \text{ Rotation (velocity) with respect to O} \]

Note: \( OX_1X_2X_3 \) fixed in the body.

Then (JNN §4.13)

• forces

\[ F_j = -\dot{U}_i m_{ji} - E_{jkl} U_i \Omega_k m_{li} \text{ with } i = 1, 2, 3, 4, 5, 6 \text{ and } j, k, l = 1, 2, 3 \]

• moments

\[ M_j = -\dot{U}_i m_{j+3,i} - E_{jkl} U_i \Omega_k m_{l+3,i} - E_{jkl} U_k U_i m_{li} \text{ with } i = 1, 2, 3, 4, 5, 6 \text{ and } j, k, l = 1, 2, 3 \]

Einstein’s \( \Sigma \) notation applies.
\[ E_{jkl} = \text{"alternating tensor"} = \begin{cases} 
0 & \text{if any } j, k, l \text{ are equal} \\
1 & \text{if } j, k, l \text{ are in cyclic order, i.e., } (1, 2, 3), (2, 3, 1), \text{or } (3, 1, 2) \\
-1 & \text{if } j, k, l \text{ are not in cyclic order i.e., } (1, 3, 2), (2, 1, 3), (3, 2, 1) 
\end{cases} \]

Note:

1. if \( \Omega_k \equiv 0 \), \( F_j = -\dot{U}_i m_{ji} \) (as expected by definition of \( m_{ij} \)). Also if \( \dot{U}_i \equiv 0 \), then \( F_j = 0 \) for any \( U_i \), no force in steady translation.

2. \( B_l \sim U_i m_{li} \) “added momentum” due to rotation of axes, \( 2) \sim \Omega \times \vec{B} \) where \( \vec{B} \) is linear momentum. (momentum from 1 coordinate into new \( x_j \) direction)

3. If \( \Omega_k \equiv 0 : M_j = -\dot{U}_i m_{j+3,i} m_{ij} - E_{jkl} U_k U_i m_{li} \)\( \text{def.of} - E_{jkl} U_k U_i m_{li} \text{even with} \dot{U}=0,M_j \neq 0 \text{due to this term} \).

Moment on a body due to pure steady translation – “Munk” moment.

Example of Munk Moment – a 2D submarine in steady translation

\[ U_1 = U \cos \theta \]
\[ U_2 = -U \sin \theta \]

Consider steady translation motion: \( \dot{U} = 0; \Omega_k = 0 \). Then
For a 2D body, $m_{3i} = m_{i3} = 0$, also $U_3 = 0$, $i, k, l = 1, 2$. This implies that:

\[ M_3 = -E_{3kl}U_iU_jm_{il} \]

\[ M_3 = -E_{312}U_1(U_1m_{21} + U_2m_{22}) - E_{321}U_2(U_1m_{11} + U_2m_{12}) \]

\[ = -U_1U_2(m_{22} - m_{11}) \]

\[ = U^2 \sin \theta \cos \theta \left( m_{22} - m_{11} \right) > 0 \]

Therefore, $M_3 > 0$ for $0 < \theta < \pi/2$ ("Bow up"). Therefore, a submarine under forward motion is unstable in pitch (yaw) (e.g., a small bow-up tends to grow with time), and control surfaces are needed:

- Restoring moment $\approx (\rho g\forall)H \sin \theta$.
- Critical speed $U_{cr}$ given by:

\[(\rho g\forall) H \sin \theta \geq U_{cr}^2 \sin \theta \cos \theta (m_{22} - m_{11})\]
Usually $m_{22} \gg m_{11}, m_{22} \approx \rho \delta$. For small $\theta$, $\cos \theta \approx 1$. So, $U_{cr}^2 \leq gH$ or $F_{cr} \equiv \frac{U_{cr}}{\sqrt{gH}} \leq 1$. Otherwise, control fins are required.