Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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Refs:
Pedlosky, *Geophysical Fluid Dynamics*
Proudman: *Dynamical Oceanography*
LaBlond and Mysak *Waves in the Ocean*
Gill: *Atmosphere-Ocean Dynamics.*

7.7 Free waves near a coast in a sea of constant depth
7-7KPwave.tex May 3, 2004

7.7.1 Governing equations

Let \( x, y \) be the horizontal coordinates where \( x \) is not necessarily from west to east. Recall the governing equations for a sea of constant depth \( H_0 \),

\[
\frac{\partial \eta}{\partial t} + H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \tag{7.7.1}
\]

\[
\frac{\partial u}{\partial t} - fv = -g \frac{\partial \eta}{\partial x} \tag{7.7.2}
\]

\[
\frac{\partial v}{\partial t} + fu = -g \frac{\partial \eta}{\partial y} \tag{7.7.3}
\]

Let us derive relations between each velocity component and the surface elevation. Differentiating (7.7.2),

\[
\frac{\partial^2 u}{\partial t^2} - f \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial t^2} - f \left[ -fu - g \frac{\partial \eta}{\partial y} \right] = -g \frac{\partial^2 \eta}{\partial x \partial t} \tag{7.7.4}
\]

Therefore,

\[
\frac{\partial^2 u}{\partial t^2} + f^2 u = -g \left( \frac{\partial^2 \eta}{\partial x \partial t} + f \frac{\partial \eta}{\partial y} \right) \tag{7.7.5}
\]

Similarly,

\[
\frac{\partial^2 v}{\partial t^2} + f^2 v = -g \left( \frac{\partial^2 \eta}{\partial y \partial t} - f \frac{\partial \eta}{\partial x} \right). \tag{7.7.6}
\]

These relations are useful for specifying boundary conditions.

Let us eliminate the velocity components to get a single equation for \( \eta \). From (7.7.5),

\[
\left( \frac{\partial^2}{\partial t^2} + f^2 \right) \left( \frac{\partial u}{\partial x} \right) = -g \left( \frac{\partial^2 \eta}{\partial x \partial t} + f \frac{\partial \eta}{\partial x \partial y} \right)
\]
and from (7.7.6),
\[
\left( \frac{\partial^2}{\partial t^2} + f^2 \right) \left( \frac{\partial v}{\partial y} \right) = -g \left( \frac{\partial^2 \eta}{\partial y^2 \partial t} - f \frac{\partial^2 \eta}{\partial x \partial y} \right)
\]
Using (7.7.1), we get
\[
-\frac{1}{H_0} \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial t^2} + f^2 \right) \eta = -g \frac{\partial}{\partial t} \nabla^2 \eta
\]
or
\[
\frac{\partial}{\partial t} \left\{ \left( \frac{\partial^2}{\partial t^2} + f^2 \right) \eta - C_0^2 \nabla^2 \eta \right\} = 0 \quad (7.7.7)
\]
This is the Klein-Gordon equation, where
\[
C_0 = \sqrt{gH_0}. \quad (7.7.8)
\]

### 7.7.2 Waves in a long channel

Consider a channel of width \( L \), \(-\infty < x < \infty, 0 < y < L \). Allowing no flux on the side walls: \( v = 0, y = 0, L \), we have, therefore, the boundary conditions,
\[
\frac{\partial^2 \eta}{\partial y \partial t} - f \frac{\partial \eta}{\partial x} = 0 \quad y = 0, L \quad (7.7.9)
\]
Consider propagating waves. Let
\[
\eta = \Re \{ \eta(y) e^{i(kx-\sigma t)} \} \quad (7.7.10)
\]
We get from (7.7.7),
\[
\frac{d^2 \eta}{dy^2} + \left[ \frac{\sigma^2 - f^2}{C_0^2} - k^2 \right] \eta = 0 \quad 0 < y < L \quad (7.7.11)
\]
and from (7.7.9),
\[
\frac{d \eta}{dy} + \frac{f k}{\sigma} \eta = 0 \quad y = 0, L. \quad (7.7.12)
\]
The general solution is
\[
\eta = A \sin \alpha y + B \cos \alpha y, \quad (7.7.13)
\]
where \( A \) and \( B \) are constants and
\[
\alpha^2 = \frac{\sigma^2 - f^2}{C_0^2} - k^2 \quad (7.7.14)
\]
Apply the boundary condition on \( y = 0 \):
\[
\alpha A + \frac{f k}{\sigma} B = 0
\]
and on $y = L$:

$$A \left( \alpha \cos \alpha L + \frac{f k}{\sigma} \sin \alpha L \right) + B \left( \frac{f k}{\sigma} \cos \alpha L - \alpha \sin \alpha L \right) = 0.$$  

For nontrivial $A$ and $B$, we must require

$$\left| \begin{array}{cc} \alpha & f k/\sigma \\ \alpha \cos \alpha L + (f k/\sigma) \sin \alpha L & (f k/\sigma) \cos \alpha L - \alpha \sin \alpha L \end{array} \right| = 0$$

This gives the eigenvalue equation,

$$\left( \sigma^2 - f^2 \right) \left( \sigma^2 - C_0^2 k^2 \right) \sin \alpha L = 0. \quad (7.7.15)$$

there are three possibilities: $\sigma = \pm f$, $\sigma = \pm k C_0$ and $\alpha = n \pi / L$.

### 7.7.3 Inertial oscillations, $\sigma^2 = f^2$

It suffices to consider $\sigma = f$. From (7.7.14),

$$\alpha^2 = -k^2 \quad (7.7.16)$$

so that

$$\frac{d^2 \eta}{dy^2} - k^2 \eta = 0 \quad 0 < y < L$$

Therefore,

$$\eta = A e^{-ky} + B e^{ky}$$

From the boundary conditions

$$\frac{d \eta}{dy} + k \eta = 0 \quad \text{at} \quad y = 0, L.$$ 

which are automatically satisfied by

$$A e^{-ky} \quad (7.7.17)$$

for any $k$. It easy to show that $B e^{ky}$ cannot satisfy both boundary conditions; we must take $B = 0$. Therefore,

$$\eta = A e^{-ky}. \quad (7.7.18)$$

Let us take a closer look at the velocity $v$. By eliminating $u$ from (7.7.1) and (7.7.2), we get

$$-\frac{1}{H} \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 v}{\partial y \partial t} = f \frac{\partial v}{\partial x} - g \frac{\partial^2 \eta}{\partial x^2}$$

Using

$$\eta = A e^{-ky} e^{ikx - ift}$$
we get
\[ \frac{\partial^2 v}{\partial y \partial t} + f \frac{\partial v}{\partial x} = gA e^{-ky} e^{ikx - i ft} \left( -k^2 + \frac{f^2}{C^2} \right) \]

Assume the solution
\[ v = V e^{-ky} e^{ikx - i ft} \]

it follows that
\[ 2ikV = gA \left( -k^2 + \frac{f^2}{C^2} \right) \]

Hence the boundary conditions at \( y = 0, L \) dictates that
\[ V = 0, \quad \text{and} \quad k^2 = \frac{f^2}{C^2} \quad (7.7.19) \]
or
\[ kR = 1 \quad (7.7.20) \]

The solution is therefore
\[ v = 0, \quad \eta = gA e^{-y/R} e^{ix/R} e^{-i ft} \quad (7.7.21) \]

From (7.7.2), we get
\[ u = \frac{gA}{C} e^{-y/R} e^{ix/R} e^{-i ft} \quad (7.7.22) \]

This is called the inertial oscillation, which is a special case of Kelvin wave.

**7.7.4 Kelvin Wave, \( \sigma = \pm C_0 k \)**

\[ \frac{\sigma}{k} = \text{phase velocity} = \pm C_0 = \pm \sqrt{gH_o} \]

This relation is the same as that for surface gravity waves. Let us focus attention to the rightward waves and take the plus sign. From Eqn. (7.7.14):
\[ \alpha^2 = -\frac{f^2}{C_0^2} = -\frac{1}{R^2} \]

where
\[ R = \frac{C_0}{f} \quad (7.7.23) \]

is defined to be the Rossby radius of deformation. Thus
\[ \alpha = \pm i \frac{\sigma}{R} f = \frac{kC_0}{f} = kR, \quad (7.7.24) \]

and the solution can be written as
\[ \bar{\eta} = Ae^{-y/R} + Be^{y/R} \quad (7.7.25) \]
Now $A e^{-y/R}$ satisfy the boundary condition (7.7.12) automatically for all $y$:

$$\left( \frac{d}{dy} + \frac{f k}{\sigma} \right) e^{-y/R} = \left( -\frac{f}{C_0} + \frac{f}{C_0} \right) e^{-y/R} = 0$$

hence $B = 0$, thus

$$\eta = A e^{-y/R} e^{i(kx - \sigma t)} = A e^{-fy/C_0} e^{i(kx - \sigma t)}. \quad (7.7.26)$$

This is the Kelvin Wave. Water is piled up along the shore $y = 0$ to the right of the wave vector. See Figure 7.7.1. A field record of English channel is in Figure 7.7.2.

There are some more peculiarities. From the momentum equations we have

$$\frac{\partial^2 v}{\partial t^2} + f^2 v = -g \left( \frac{\partial^2 \eta}{\partial y \partial t} - f \frac{\partial \eta}{\partial x} \right)$$
Figure 7.7.2: Kelvin wave in British Channel.

\[ = -g \left( Ae^{ikx-i\sigma t}e^{-fy/C_0} \right) \left[ -\frac{f}{C_0} (-i\sigma_0) - fik \right] \]

\[ = -gAe^{ikx}e^{-fy/C_0} \frac{f\sigma}{C_0} (1 - 1) = 0. \]

Therefore,

\[ v = 0 \quad (7.7.27) \]

identically. Now the x-momentum equation and mass conservation equation reduce to

\[ \frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x} \quad (7.7.28) \]

and

\[ \frac{\partial \eta}{\partial t} = -H_0 \frac{\partial u}{\partial x} \quad (7.7.29) \]

These are formally the long wave equation in one space dimension \( x \). But \( \eta \) and \( u \) depend on \( y \)! Indeed for the propagating wave

\[ \frac{\partial u}{\partial t} = -i\sigma u = -g \frac{\partial \eta}{\partial x} = -ikg\eta \quad \sigma = C_0k \]

hence

\[ u = \frac{kg}{\sigma} \eta \]
From the solution (7.7.26),

\[
\frac{\partial \eta}{\partial y} = -f \frac{\eta}{C_0} = -f \frac{\sigma u}{g k} = -f \frac{C_0}{g}.
\]

Therefore,

\[
u = -\frac{g}{f} \frac{\partial \eta}{\partial y}
\]

This is a state of Quasi-static Geostrophy!

Note

\[
R = \text{Rossby radius of deformation} = \frac{C_0}{f} = \left[ \frac{\text{vel}}{1/\text{Time}} \right] = [\text{Length}]
\]

If \(H_0 = 30m\)

\[
C_0 = \sqrt{gH_0} = 1.732 \times 10m/s, \quad f = 10^{-4} s^{-1}
\]

\[
R = \frac{C_0}{f} = 1.732 \times 10^5 m = 173 km.
\]

If \(H_0 = 1000m\)

\[
R = 1000 km.
\]

**7.7.5 Poincare waves**

Consider the eigenvalue condition,

\[
\sin \alpha L = 0 \quad \alpha = \alpha_n = \frac{n\pi}{L} \quad n = 0, 1, 2, 3, \cdots
\]

(7.7.30)

From Eqn. (4.3)

\[
\frac{\sigma_n^2 - f^2}{C_0^2} - k^2 = \left( \frac{n\pi}{L} \right)^2
\]

or

\[
\sigma_n = \pm \left\{ f^2 + C_0^2 \left[ k^2 + \left( \frac{n\pi}{L} \right)^2 \right] \right\}^{1/2}.
\]

This relation between frequency and wave number \(\sigma = \sigma(k)\) is called the dispersion relation.

The dispersion relation can also be written

\[
\frac{\sigma_n}{f} = \pm \sqrt{1 + \left( \frac{C_0}{f} \right)^2 \left[ k^2 + \left( \frac{n\pi}{L} \right)^2 \right]} = \pm \sqrt{1 + \left[ (kR)^2 + \left( \frac{n\pi R}{L} \right)^2 \right]}.
\]

(7.7.31)

See Figure 7.7.3
The free surface of the \( n \)-th Poincare mode is:

\[
\eta_n = B \left( \frac{A}{B} \sin \alpha_n y + \cos \alpha_n y \right) \\
= B \left( -\frac{f k}{\sigma \alpha_n} \sin \alpha_n y + \cos \alpha_n y \right) \\
= B \left( \cos \frac{n \pi y}{L} - \frac{L}{n \pi} \frac{f k}{\sigma_n} \sin \frac{n \pi y}{L} \right).
\]

or,

\[
\eta_n = \eta_0 \left( \cos \frac{n \pi y}{L} - \frac{L}{n \pi} \frac{f k}{\sigma_n} \sin \frac{n \pi y}{L} \right) \cos (kx - \sigma_n t) \tag{7.7.32}
\]

The velocity components are given by

\[
u = \eta_0 \left( \frac{C_0^2 k}{\sigma_n} \cos \frac{n \pi y}{L} - \frac{f L}{n \pi} \sin \frac{n \pi y}{L} \right) \cos (kx - \sigma_n t) \tag{7.7.33}
\]

\[
v = -\eta_0 \frac{L}{H_0 \sigma_n n \pi} \left( f^2 + \frac{C_0^2 n^2 \pi^2}{L^2} \right) \sin \frac{n \pi y}{L} \sin (kx - \sigma_n t) \tag{7.7.34}
\]

Additional types of waves exist if the depth is not constant, or the water is stratified.
Figure 7.7.3: Dispersion relation between frequency and wave number.

Figure 3.9.2: The dispersion diagram for Poincaré and Kelvin waves, showing the coincidence of the inertial oscillation $\sigma f = \pm 1$ and the Kelvin mode at $kR = 1$. 