7.2 Taylor-Proudman theorem and Vorticity in inviscid rotating fluids

We first show that in a steady rotating flow of inviscid and homogeneous fluid, if the Rossby number is small, then the flow is essentially two-dimensional. This is known as the Taylor-Proudman theorem.

Under these conditions, the momentum equation reads,

$$2\vec{\Omega} \times \vec{q} = -\nabla \frac{p}{\rho} \tag{7.2.1}$$

Taking the curl of both sides we get

$$\nabla \times (\vec{\Omega} \times \vec{q}) = 0 \tag{7.2.2}$$

Using the identity

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A} \nabla \cdot \vec{B} - \vec{B} \nabla \cdot \vec{A} + \vec{B} \cdot \nabla \vec{A} - \vec{A} \cdot \nabla \vec{B} \tag{7.2.3}$$

we get

$$\vec{\Omega} \nabla \cdot \vec{q} - \vec{q} \nabla \cdot \vec{\Omega} + \vec{q} \cdot \nabla \vec{\Omega} - \vec{\Omega} \cdot \nabla \vec{q} = 0$$

Invoking continuity and the constancy of $\vec{\Omega}$ we obtain

$$\vec{\Omega} \cdot \nabla \vec{q} = 0 \tag{7.2.4}$$

Thus the velocity field does not vary in the direction of $\vec{\Omega}$, say $z$. Note that $\vec{q}$ can still have three components, but they must all be independent of $z$. This is the

**Theorem 1** Taylor-Proudman theorem: A steady and slow flow in a rotating fluid is two-dimensional in the plane perpendicular to the vector of angular velocity.

Laboratory verification has been demonstrated in a setup shown in figure 7.2.1.

More generally, let us consider the vorticity transport in a rotating and inviscid fluid. Let $\vec{\zeta} = \nabla \times \vec{q}$ and use the identity

$$\vec{\zeta} \times \vec{q} = \vec{q} \cdot \nabla \vec{q} - \nabla \frac{|\vec{q}|^2}{2}$$
Figure 7.2.1: Taylor’s experiment showing the Taylor column above a truncated cylinder in a rotating fluid. The large container with water rotates but the cylinder is fixed in space. From Kundu.

The momentum equation can be written:

\[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{\zeta} \times \mathbf{q} + 2\mathbf{\Omega} \times \mathbf{q} = -\nabla p \rho + \nabla \left( \phi - \frac{|\mathbf{q}|^2}{2} \right) \]  

Taking the curl of the above equation:

\[ \frac{\partial \mathbf{\zeta}}{\partial t} + \nabla \times \left( (2\mathbf{\Omega} + \mathbf{\zeta}) \times \mathbf{q} \right) = \frac{\nabla \rho \times \nabla p}{\rho^2} \]

Using the identity (7.2.3), we get

\[ \nabla \times \left( (2\mathbf{\Omega} + \mathbf{\zeta}) \times \mathbf{q} \right) = -q \nabla \cdot (2\mathbf{\Omega} + \mathbf{\zeta}) + (2\mathbf{\Omega} + \mathbf{\zeta}) \nabla \cdot q + \mathbf{q} \cdot \nabla (2\mathbf{\Omega} + \mathbf{\zeta}) - (2\mathbf{\Omega} + \mathbf{\zeta}) \cdot \nabla q \]
The first term on the right vanishes because \( \vec{\Omega} = \text{constant} \) and the divergence of curl is zero; the second vanishes for incompressible fluids. Let \( \vec{\zeta}_a = \vec{\zeta} + 2\vec{\Omega} \) = absolute vorticity

\[
\frac{D\vec{\zeta}}{Dt} = \frac{\partial \vec{\zeta}}{\partial t} + \vec{q} \cdot \nabla \vec{\zeta} = \vec{\zeta}_a \cdot \nabla \vec{q} + \frac{\nabla \rho \times \nabla p}{\rho^2}
\]

(7.2.6)

In a fluid of constant density and a steady flow of small Rossby number

\[
\epsilon = \text{Rossby No.} = \frac{u}{2\Omega L} \ll 1
\]

then

\[
\frac{\zeta}{2\Omega} \approx \frac{u}{2\Omega L} \ll 1
\]

(7.2.6) reduces to (7.2.4).