6.5 Geothermal Plume

Consider a steady, two dimensional plume due to a source of intense heat in a porous medium. From Darcy’s law:

\[ \frac{\mu}{k} u = -\frac{\partial p}{\partial x} \]  
(6.5.1)

where \( k \) denotes the permeability, and

\[ \frac{\mu}{k} w = -\frac{\partial p}{\partial z} - \rho g \]  
(6.5.2)

These are the momentum equations for slow motion in porous medium. Mass conservation requires

\[ u_x + w_z = 0 \]  
(6.5.3)

Energy conservation requires

\[ u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \]  
(6.5.4)

where

\[ \alpha = \frac{K}{\rho_0 C} \]  
(6.5.5)

denotes the thermal difusivity.

Equation of state:

\[ \rho = \rho_0 (1 - \beta(T - T_0)) \]  
(6.5.6)

Consider the flow induced by a strong heat source. Let

\[ T - T_0 = T', \quad p = p_o + p' \]

where \( p_o \) is the hydrostatic pressure satisfying

\[ -\frac{\partial p_o}{\partial z} - \rho_0 g = 0. \]

Eqn. (6.5.2) can be written

\[ \frac{\mu}{k} w = -\frac{\partial p'}{\partial z} + g \rho_0 \beta T'. \]  
(6.5.7)
6.5.1 Boundary layer approximation

Eliminating $p'$ from Eqns. (6.5.7) and (6.5.1), we get

$$\frac{\mu}{k} (w_x - u_z) = g \rho_0 \beta T_x'. $$

Let $\psi$ be the stream function such that

$$u = \psi_z, \quad w = -\psi_x$$

then

$$\psi_{xx} + \psi_{zz} = -\frac{g \rho_0 \beta k}{\mu} T_x'. \quad (6.5.8)$$

For an intense heat source, we expect the plume to be narrow and tall. Let us apply the boundary layer approximation and check its realm of validity later,

$$u \ll w, \quad \frac{\partial}{\partial x} \gg \frac{\partial}{\partial z}.$$  

hence

$$\psi_{xx} \simeq -\frac{\rho_0 \beta k}{\mu} T_x' \quad (6.5.9)$$

which is the same as ignoring $\partial p'/\partial z$ in Eqn. (6.5.7).

This can be confirmed since $u \ll w \ \partial p'/\partial x \approx 0, \ p'$ inside the plume is the same as that outside the plume. But

$$\frac{\partial p'}{\partial z} = 0$$

outside the plume, hence $\partial p'/\partial z \approx 0$ inside as well.

Applying the B.L. approximation to Eqn. (6.5.4)

$$u T'_x + w T'_z = \alpha T''_{xx} \quad (6.5.10)$$

Using the continuity equation we get

$$(u T')_x + (w T')_z = \alpha T''_{xx}.$$  

Integrating across the plume,

$$\frac{\partial}{\partial z} \int_{-\infty}^{\infty} T' \, dx = 0 \quad (6.5.11)$$

since $T' = 0$ outside the plume. It follows that

$$\rho_0 C \int_{-\infty}^{\infty} w T' \, dx = -\rho_0 C \int_{-\infty}^{\infty} \psi_x T' \, dx = Q = \text{constant}. \quad (6.5.12)$$
6.5.2 Normalization

Let us take
\[ x = B\bar{x}, \quad z = H\bar{z}, \quad u = \frac{WB}{H}\bar{u}, \quad w = W\bar{w}, \quad T' \rightarrow \Delta T\theta \] (6.5.13)
where \(H, B, \Delta T\) and \(W\) are to be determined to get maximum simplicity. We then get from the momentum equation,
\[ \bar{w} = \bar{\psi}_x = -\frac{g\rho_0\beta\Delta T}{\mu W} \theta, \]
from the energy equation,
\[ \bar{u}\theta_x + \bar{w}\theta_z = \frac{\alpha H}{WB^2}\theta_{\bar{z}\bar{z}}, \]
and from the total flux condition,
\[ \rho_0 CWB\Delta \int_{-\infty}^{\infty} \bar{w}\theta d\bar{x} = Q \]
Let us choose
\[ \frac{g\rho_0\beta\Delta T}{\mu W} = 1 \] (6.5.14)
\[ \frac{\alpha H}{WB^2} = 1 \] (6.5.15)
and
\[ \rho_0 CWB\Delta T = Q, \] (6.5.16)
which gives three relations among four scales, \(B, H, W, \Delta T\). Then
\[ \bar{w} = \bar{\psi}_x = -\theta, \] (6.5.17)
from the energy equation,
\[ \bar{u}\theta_x + \bar{w}\theta_z = \theta_{\bar{z}\bar{z}}, \] (6.5.18)
and from the total flux condition,
\[ \int_{-\infty}^{\infty} \bar{w}\theta d\bar{x} = 1 \] (6.5.19)
In addition we require that
\[ w(\pm \infty, z) = 0, \quad \theta(\pm \infty, z) = 0 \] (6.5.20)
\[ u(0, z) = \frac{\partial w(0, z)}{\partial x} = 0, \quad x = 0. \] (6.5.21)
From here on we omit overhead bars in all dimensionless equations for brevity.
6.5.3 Similarity solution

Now let
\[ x = \lambda^a x^* \quad z = \lambda^b z^* \quad \psi = \lambda^c \psi^* \quad \theta = \lambda^d \theta^*. \]

From Eqn. (6.5.17)
\[ \lambda^{c-a} \left( \frac{\partial \psi^*}{\partial x^*} \right) = -\lambda^d \theta^*. \]

For invariance we require,
\[ c - a = d. \quad (6.5.22) \]

From (6.5.19)
\[ - \int \frac{\partial \psi^*}{\partial x^*} dx^* \lambda^{c-a+a+d} = 1. \]

therefore,
\[ a + d = 0. \quad (6.5.23) \]

From Eqn. (6.5.18)
\[ \lambda^{c+d-a-b} = \lambda^{d-2a}. \]

implying,
\[ c + a - b = 0. \quad (6.5.24) \]

Finally
\[ c = \frac{a}{2}, \quad d = -\frac{a}{2}, \quad b = \frac{3}{2} a. \]

In view of these we introduce the following similarity variables,
\[ \eta = \frac{x}{z^{2/3}}, \quad \psi = z^{1/3} f(\eta), \quad \theta = z^{-1/3} h(\eta). \quad (6.5.25) \]

Note that at the center line \( \eta = 0 \)
\[ w = -\psi_x \propto z^{1/3} f'(0)(-z^{-2/3}) \sim z^{-1/3} f'(0) \sim z^{-1/3} \quad (6.5.26) \]
\[ \theta \propto z^{-1/3} h(0) \quad (6.5.27) \]

and
\[ b \propto z^{2/3} \quad (6.5.28) \]

Thus the velocity and temperature along the centerline decay as \( z^{-1/3} \) and the plume width grows as \( z^{2/3} \).

Substituting these into Eqns. (6.5.17) and (6.5.18), we get, after some algebra
\[ -\frac{df}{d\eta} = h \quad (6.5.29) \]

and
\[ \frac{d}{d\eta} (fh) = 3 \frac{d^2 h}{d\eta^2}. \quad (6.5.30) \]
The boundary conditions are,

\[ f = 0 \quad (\psi = 0) \]
\[ f''(0) = 0, \quad (w(0, z) = w_{max}) \]
\[ f(\pm \infty), \quad f'(\pm \infty) = 0 \]
\[ h(\pm \infty) = 0. \]

Integrating Eqn. (6.5.30), we get

\[ fh = 3h'. \]

Using Eqn. (6.5.29), we get

\[ ff' = 3f''. \]

Integrating again, we get

\[ -6f' = f_0^2 - f^2 \]

where \( f_0 = f_{max}. \) Let \( f = -f_0 F, \) then

\[ f_0(1 - F^2) = 6F'', \text{ or } \frac{dF}{1 - F^2} = \frac{f_0 d\eta}{6} \]

which can be integrated to give

\[ \frac{f_0 \eta}{6} = \frac{1}{2} \ln \frac{1 + F}{1 - F} \]

Thus

\[ \left( \frac{1 + F}{1 - F} \right)^{1/2} = e^{f_0 \eta/6} \]

or

\[ \left( \frac{1 + F'}{1 - F} \right) = e^{f_0 \eta/3} \]

Solving for \( F, \) we get

\[ F = \frac{e^{f_0/3} - 1}{e^{f_0/3} + 1} = \tanh \frac{f_0 \eta}{6} \quad (6.5.31) \]

What is \( f_0? \) Let us use Eqn. (6.5.29)

\[ - \int_{-\infty}^{\infty} \frac{df}{d\eta} h \, d\eta = \int_{-\infty}^{\infty} (f')^2 d\eta = 1 \]

since

\[ f' = -f_0 F' = -\frac{f_0^2}{6} \text{sech}^2 \frac{f_0 \eta}{6} \]

and

\[ h = -f'. \]
Therefore,
\[
\left( \frac{f_0^2}{6} \right)^2 \int_{-\infty}^{\infty} \text{sech}^4 \left( \frac{f_0 \eta}{6} \right) \, d\eta = \frac{f_0^3}{6} \int_{-\infty}^{\infty} \text{sech}^4 \zeta \, d\zeta = 1.
\]

Since
\[
\int_{-\infty}^{\infty} \text{sech}^4 z \, dz = 4/3.
\]
we get \( f_0 \)!
\[
f_0 = \left( \frac{9}{2} \right)^{1/3}
\]
(6.5.32)

The solution is
\[
f = \left( \frac{9}{2} \right)^{1/3} \tanh \left( \frac{9}{2} \right)^{1/3} \frac{\eta}{6}
\]
(6.5.33)
and
\[
h = -f' = -\left( \frac{9}{2} \right)^{2/3} \text{sech}^2 \left( \frac{9}{2} \right)^{1/3} \frac{\eta}{6}
\]
(6.5.34)

Computed results are given in Figures.

**Remark** Checking the boundary layer approximation.

\[
\frac{\partial^2 \psi}{\partial x^2} \sim z^{-1}, \quad \frac{\partial^2 \psi}{\partial z^2} \sim z^{-5/3}
\]
\[
\frac{\partial^2 T'}{\partial x^2} \sim z^{-5/3}, \quad \frac{\partial^2 T'}{\partial z^2} \sim z^{-7/3}
\]
hence for large \( z \), B. L. approximation is good.

**6.5.4 Return to physical coordinates**

Start from
\[
\eta = \frac{x}{z^{2/3}}
\]
(6.5.35)
\[
\frac{\bar{\psi}}{z^{1/3}} = f(\eta)
\]
(6.5.36)
\[
z^{1/3} \theta = h(\eta)
\]
(6.5.37)

Then
\[
\eta = \frac{x/B}{(z/H)^{2/3}} = \left( \frac{H^{2/3}}{B} \right) \left( \frac{x}{z^{2/3}} \right)
\]
(6.5.38)

By eliminating \( H \) and \( \Delta T \) from (6.5.35) and (6.5.37), we get
\[
W = \sqrt{\frac{Qg\beta}{CB}}
\]
Figure 6.5.1: Theoretical solution for a geothermal plume due to Yih

From (6.5.36), we get

\[
\frac{H}{B^2} = \frac{W}{\alpha} = \frac{1}{\alpha} \sqrt[3]{\frac{Q g \beta}{C B}}
\]

It follows that

\[
\frac{H}{B^{3/2}} = \frac{1}{\alpha} \sqrt[3]{\frac{Q g \beta}{C}} \tag{6.5.39}
\]

Now

\[
\frac{\bar{\psi}}{z^{1/3}} = \frac{\psi}{W B} \left( \frac{z}{H} \right)^{-1/3} = \left( \frac{H^{1/3}}{W B} \right) \left( \frac{\psi}{z^{1/3}} \right) \tag{6.5.40}
\]
It can be shown that

\[ \frac{H^{1/3}}{WB} = \frac{1}{\sqrt[3]{Qg\\beta}} \left( \frac{H}{B^{3/2}} \right)^{1/3} = \frac{1}{\alpha^{1/3}} \left( \frac{C}{Qg\\beta} \right)^{1/3} \]

which depends on the fluid properties and the given heat source strength.

Also

\[ z^{1/3} = h(\eta) = (Hz)^{1/3} \Delta TT^m = (H^{1/3} \Delta T) z^{1/3} T^r \]

We can show that

\[ H^{1/3} \Delta T = \frac{1}{\nu} \frac{1}{\sqrt{g\\beta C}} \left( \frac{1}{\alpha \sqrt{Qg\\beta}} \right)^{1/3} = \frac{Q^{1/6}}{\nu(\alpha g\\beta)^{1/3} C^{2/3}} \]
which also depends on the fluid properties and the given heat source strength.