3.9 Impulsive motion of a blunt body and tendency for separation

Ref: H. Schlichting, Boundary layer theory, p 400 ff.

As an example of unsteady boundary layer, let us consider the initial stage \((U_o T/L \ll 1)\) of a boundary layer due to the impulsive start of motion near a blunt body, see the sketch in Figure 3.9.1.

Let us start with the boundary layer approximation and introduce a perturbation expansion in powers of the small ratio \(U_o T/L\),

\[
\begin{align*}
    u &= u^{(1)} + \left(\frac{U_o T}{L}\right) u^{(2)} + \left(\frac{U_o T}{L}\right)^2 u^{(3)} + \cdots, \\
p &= p^{(1)} + \left(\frac{U_o T}{L}\right) p^{(2)} + \left(\frac{U_o T}{L}\right)^2 p^{(3)} + \cdots
\end{align*}
\]

We then get

\[
\begin{align*}
    u_x^{(1)} + v_y^{(1)} + \left(\frac{U_o T}{L}\right) \left(u_x^{(2)} + v_y^{(2)}\right) + \cdots &= 0,
\end{align*}
\]

Figure 3.9.1: Boundary layer around a blunt body
and

\[ u_t^{(1)} + \frac{U_o T}{L} u_t^{(2)} + \frac{U_o T}{L} (u^{(1)} u_x^{(1)} + v^{(1)} u_y^{(1)}) + O \left( \frac{U_o T}{L} \right)^2 = \frac{U_o T}{L} U U_x + u^{(1)}_{yy} + \frac{U_o T}{L} u^{(2)}_{yy} + O \left( \frac{U_o T}{L} \right)^2 \]  

(3.9.4)

(3.9.5)

Equating the coefficients of \( \left( \frac{U_o T}{L} \right)^0 \) we get the first (leading) order perturbation equations in normalized coordinates,

\[ u_x^{(1)} + v_y^{(1)} = 0, \]

(3.9.6)

\[ u_t^{(1)} = u_{yy} \]

(3.9.7)

subject to the initial conditions:

\[ u^{(1)} = v^{(1)} = 0, \quad t = 0, \quad \forall y; \]

(3.9.8)

and the boundary conditions

\[ u^{(1)} = v^{(1)} = 0, \quad y = 0, \quad \forall t; \]

(3.9.9)

\[ u^{(1)} = U, \quad y \to \infty \]

(3.9.10)

Equating the coefficient of \( \left( \frac{U_o T}{L} \right) \), we get the second order perturbation equations in normalized coordinates,

\[ u_x^{(2)} + v_y^{(2)} = 0, \]

(3.9.11)

\[ u_t^{(2)} + (u^{(1)} u_x^{(1)} + v^{(1)} u_y^{(1)}) = U U_x + u^{(2)}_{yy} + O \left( \frac{U_o T}{L} \right)^2 \]  

(3.9.12)

subject to the same initial and boundary conditions on the wall as the first order problem, except that

\[ u^{(2)} \to 0, \quad y \to \infty \]

(3.9.13)

To return to physical variables, we need only add the coefficient \( \nu \) in front of the viscous stress term \( u_{yy} \) in (3.9.7), and (3.9.12). The first order problem for the tangential velocity is precisely the Rayleigh problem

\[ u_t^{(1)} = u_{yy} \]

(3.9.14)

subject to the initial conditions:

\[ u^{(1)} = 0, \quad t = 0, \quad \forall y; \]

(3.9.15)

and the boundary conditions

\[ u^{(1)} = 0, \quad y = 0, \quad \forall t; \]

(3.9.16)
\[ u^{(1)} = U, \quad y \to \infty \] (3.9.17)

The solution is
\[ u^{(1)}(x, y, t) = U(x) \text{erf}(\eta) = U(x) \frac{2}{\sqrt{\pi}} \int_{0}^{\eta} e^{-\eta^2} \, d\eta \] (3.9.18)

where
\[ \eta = \frac{y}{\sqrt{2} \nu t} \] (3.9.19)

Integrating the continuity equation (3.9.6) we get
\[ v^{(1)} = -\int_{0}^{y} \frac{\partial u}{\partial x} \, dy = -\frac{dU}{dx} 2\sqrt{\nu t} \int_{0}^{\eta} \text{erf}(\eta) \, d\eta \] (3.9.20)

To simply the notation we introduce
\[ \text{erf}(\eta) = \zeta'_{0}(\eta), \quad \int_{0}^{\eta} \text{erf}(\eta) \, d\eta = \zeta_{0}(\eta) \] (3.9.21)

so that
\[ u^{(1)} = U(x)\zeta'_{0}(\eta), \quad v^{(1)} = -\frac{dU}{dx} 2\sqrt{\nu t} \zeta_{0}(\eta) \] (3.9.22)

The second-order approximation is
\[ u^{(2)}_t - \nu u^{(2)}_{yy} = UU_x \left[ 1 - (\text{erf}(\eta))^2 + e^{-\eta^2} \int_{0}^{\eta} \text{erf}(\eta) \, d\eta \right] \]
\[ = UU_x \left[ 1 - (h')^2 + hh'' \right] = UU_x F(\eta) \] (3.9.23)

subject to the initial and boundary conditions that
\[ u^{(2)}(y, 0) = 0, \quad u^{(2)}(y, t) = 0 \quad \text{for} \ y = 0, \infty \] (3.9.24)

The right hand side of (3.9.23) can be worked out so that
\[ u^{(2)}_t - \nu u^{(2)}_{yy} = UU_x \left[ 1 - (\text{erf}(\eta))^2 + e^{-\eta^2} \int_{0}^{\eta} \text{erf}(\eta) \, d\eta \right] \]
\[ = UU_x \left[ 1 - (h')^2 + hh'' \right] = UU_x F(\eta) \] (3.9.25)

A similarity solution is possible. Let us seek a one-parameter transformation,
\[ u^{(2)} = \lambda^a u^{(2)}', \quad t = \lambda^b t', \quad y = \lambda^c y' \]

From (3.9.23) we get
\[ \lambda^{a-b} \frac{\partial u^{(2)}'}{\partial t'} - \nu \lambda^{a-2c} \frac{\partial^2 u^{(2)}'}{\partial y'^2} = UU_x F(\lambda^{c-b/2} \eta') \]

Note that \( x \) is just a parameter. Clearly \( a = b = 2c \) so that we can take
\[ \frac{u^{(2)}}{t} = f(\eta)UU_x \] (3.9.26)
Substituting (3.9.26) into (3.9.25), we get a linear ordinary differential equation

\[ f'' + 2\eta f' - 4f = 4\left[ (\zeta_0')^2 - \zeta_0\zeta_0'' - 1 \right] \quad (3.9.27) \]

subject to the boundary conditions that

\[ f = 0, \quad \eta = 0, \infty \quad (3.9.28) \]

The solution is not difficult (see Schlichting, eq. 15.43, p. 400).

\[
f = \operatorname{erfc}(\eta) \left[ -\frac{3}{\sqrt{\pi}} e^{-\eta^2} + 2 - \left( \frac{4}{3\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}} \right) \right] + \frac{1}{2} (2\eta^2 - 1) \operatorname{erfc}^2(\eta) + \frac{2}{3} e^{-2\eta^2} + e^{-\eta^2} \left[ \frac{\eta}{\sqrt{\pi}} - \frac{4}{3\pi} + \eta \left( \frac{3}{\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}} \right) \right] \quad (3.9.29) \]

The solution is plotted in Figure 3.9.2.

The total solution is

\[ u = U \operatorname{erf}(\eta) + tUU_x f(\eta) \quad (3.9.30) \]

Figure 3.9.2: Solution to the problem of impulsive start.

**Separation**

For a given \( U(x) \) when and where will separation first occur? Namely, when is

\[ \frac{\partial u}{\partial y} = 0 \text{ at } y = 0 \]
Let us use (3.9.30) for a crude estimate. Since
\[ \frac{\partial u}{\partial y} = [U'(\text{erf} \eta) + UU_x t f'(\eta)] \frac{\partial \eta}{\partial y} \]

It can be shown that at \( \eta = 0 \),
\[ (\text{erf} \eta)' = \frac{2}{\sqrt{\pi}}, \quad f'(\eta) = \frac{2}{\sqrt{\pi}} \left( 1 + \frac{4}{3\pi} \right) \]

It follows that
\[ U + t_s \left( 1 + \frac{4}{3\pi} \right) UU_x = 0 \]
or
\[ t_s = \frac{0.7}{UU_x} \]  

(3.9.31)

Note that \( t_s > 0 \) only for \( U_x < 0 \), i.e., a decelerated flow. This is a very crude and mathematically illegitimate estimate since we are equating two terms of different order.

Nevertheless let us apply this result to the impulsive flow passing a circular cylinder from the left. Let \( U_o \) be the constant velocity at infinity and the polar angle \( \theta \) be measured from the upstream stagnation point, then \( x = a\theta \) where \( a \) is the radius, see Figure 3.9.3. It is well known in the potential theory that the potential is
\[ \phi = U_o \left( r + \frac{a^2}{r} \right) \cos(\pi - \theta) \]

The tangential velocity along the cylinder \( r = a \) is
\[ \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{U_o}{r} \left( r + \frac{a^2}{r} \right) \sin(\pi - \theta), \quad r = a \]
or
\[ U = 2U_o \sin(\pi - \theta) = 2U_o \sin(\theta) = 2U_o \sin x/a \]

The minimum \( t_s \) occurs at the rear stagnation point, \( x = \pi a \) at which
\[ t_s = \frac{0.35a}{U_o}, \quad \text{or} \quad \frac{U_ot_s}{a} = 0.35 \]

Note that the last condition indicates the illegitimacy of this estimate. Nevertheless we use it here as an order-of-magnitude guide which may be improved by working out higher order terms.

In offshore structures, wave induced oscillatory flows around a pile can be separated and hence affect the drag force on the pile. As an order estimate we take \( U_o = \omega A \) where \( \omega = \text{frequency} \) and \( A = \text{wave amplitude} \). Hence there is no separation if
\[ \frac{\omega At_s}{a} < 0.35, \quad \text{or} \quad \frac{A}{a} < \frac{0.35}{\omega t_s} \]
Since flow changes direction after every half period $\pi/\omega$, there is no separation in every half period if

$$\frac{A}{a} < \frac{0.35}{\pi} = 0.1$$

This is of course very crude. Experimentally Keulegan and Carpenter have established that separation occurs in waves if $A/a$ exceeds 1. The ratio $A/a$ is now known as the Keulegan and Carpenter number.

![Figure 3.9.3: Definition of coordinates for a circular cylinder.](image)