LECTURE 2: Stress Conditions at a Fluid-fluid Interface

We proceed by deriving the normal and tangential stress boundary conditions appropriate at a fluid-fluid interface characterized by an interfacial tension $\sigma$.

Consider an interfacial surface $S$ bound by a closed contour $C$ (Figure 1). One may think of there being a force per unit length of magnitude $\sigma$ in the $s$-direction at every point along $C$ that acts to flatten the surface $S$. Perform a force balance on a volume element $V$ enclosing the interfacial surface $S$ defined by the contour $C$:

$$\int_V \rho \frac{Du}{Dt} \, dV = \int_V f \, dV + \int_S \left[ t(\mathbf{n}) + \hat{t}(\hat{\mathbf{n}}) \right] \, dS + \int_C \sigma s \, dl$$

Here $\ell$ indicates arclength and so $dl$ a length increment along the curve $C$. $t(\mathbf{n}) = \mathbf{n} \cdot \mathbf{T}$ is the stress vector, the force/area exerted by the upper (+) fluid on the interface. The stress tensor is defined in terms of the local fluid pressure and velocity field as $\mathbf{T} = -p \, \mathbf{I} + \mu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]$. Similarly, the stress exerted on the interface by the lower (-) fluid is $\hat{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \hat{\mathbf{T}} = -\mathbf{n} \cdot \hat{\mathbf{T}}$ where $\hat{\mathbf{T}} = -\hat{p} \, \mathbf{I} + \hat{\mu} \left[ \nabla \hat{\mathbf{u}} + (\nabla \hat{\mathbf{u}})^T \right]$.

Physical interpretation of terms

$$\int_V \rho \frac{Du}{Dt} \, dV$$: inertial force associated with acceleration of fluid within $V$

$$\int_V f \, dV$$: body forces acting on fluid within $V$

$$\int_S t(\mathbf{n}) \, dS$$: hydrodynamic force exerted at interface by fluid +

$$\int_S \hat{t}(\hat{\mathbf{n}}) \, dS$$: hydrodynamic force exerted at interface by fluid -

$$\int_C \sigma s \, dl$$: surface tension force exerted along perimeter $C$

Figure 1: A surface $S$ and bounding contour $C$ on an interface between two fluids. The upper fluid (+) has density $\rho$ and viscosity $\mu$; the lower fluid (-), $\hat{\rho}$ and $\hat{\mu}$. $\mathbf{n}$ represents the unit outward normal to the surface, and $\mathbf{n} = -\mathbf{n}$ the unit inward normal. $\mathbf{m}$ the unit tangent to the contour $C$ and $s$ the unit vector normal to $C$ but tangent to $S$. 
Now if $\epsilon$ is the typical lengthscale of the element $V$, then the acceleration and body forces will scale as $\epsilon^3$, but the surface forces will scale as $\epsilon^2$. Hence, in the limit of $\epsilon \to 0$, we have that the surface forces must balance:

$$\int_S [t(n) + \hat{t}(\hat{n})] \, dS + \int_C \sigma \, d\ell = 0$$

Now we have that

$$t(n) = n \cdot T, \quad \hat{t}(n) = \hat{n} \cdot \hat{T} = -n \cdot \hat{T}$$

Moreover, the application of Stokes Theorem (see Appendix A) allows us to write

$$\int_C \sigma s \, d\ell = \int_S \nabla_s \sigma - \sigma n (\nabla \cdot n) \, dS$$

where the tangential gradient operator, defined by

$$\nabla_s = [I - nn] \cdot \nabla = \nabla - n \frac{\partial}{\partial n}$$

appears because $\sigma$ and $n$ are defined only on the surface. We proceed by dropping the subscript $s$ on $\nabla$, with this understanding.

The surface force balance thus becomes:

$$\int_S [n \cdot T - n \cdot \hat{T}] \, dS = \int_S \sigma n (\nabla \cdot n) - \nabla \sigma \, dS$$

(1)

Now since the surface element is arbitrary, the integrand must vanish identically. One thus obtains the interfacial stress balance equation.
The jump in normal stress across the interface must balance the curvature force per unit area. We note that a surface with non-zero curvature ($\nabla \cdot \mathbf{n} \neq 0$) reflects a jump in normal stress across the interface.

**Tangential Stress Balance**

Taking $\mathbf{t} \cdot (2)$, where $\mathbf{t}$ is any unit vector tangent to the interface, yields the tangential stress balance equation:

$$\mathbf{t} \cdot \nabla \sigma$$
balance at the interface:

\[ n \cdot T \cdot t - n \cdot \hat{T} \cdot t = \nabla \sigma \cdot t \]

(4)

Physical Interpretation:

- the LHS represents the jump in tangential components of the hydrodynamic stress at the interface
- the RHS represents the tangential stress associated with gradients in \( \sigma \), as may result from gradients in temperature or chemical composition at the interface
- the LHS contains only velocity gradients, not pressure; therefore, a non-zero \( \nabla \sigma \) at a fluid interface must always drive motion.

Appendix A

Recall Stokes Theorem:

\[ \int_C \mathbf{F} \cdot d\ell = \int_S \mathbf{n} \cdot (\nabla \wedge \mathbf{F}) \, dS \]

Along the contour \( C \), \( d\ell = \mathbf{m} \, d\ell \), so that we have

\[ \int_C \mathbf{F} \cdot \mathbf{m} \, d\ell = \int_S \mathbf{n} \cdot (\nabla \wedge \mathbf{F}) \, dS \]

Now let \( \mathbf{F} = f \wedge \mathbf{b} \), where \( \mathbf{b} \) is an arbitrary constant vector. We thus have

\[ \int_C (f \wedge \mathbf{b}) \cdot \mathbf{m} \, d\ell = \int_S \mathbf{n} \cdot (\nabla \wedge (f \wedge \mathbf{b})) \, dS \]

Now use standard vector identities to see:

\[ (f \wedge \mathbf{b}) \cdot \mathbf{m} = -\mathbf{b} \cdot (f \wedge \mathbf{m}) \]

\[ \nabla \wedge (f \wedge \mathbf{b}) = f(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot f) + \mathbf{b} \cdot \nabla f - f \cdot \nabla \mathbf{b} \]

\[ = -\mathbf{b}(\nabla \cdot f) + \mathbf{b} \cdot \nabla f \]

since \( \mathbf{b} \) is a constant vector. We thus have
\[ \mathbf{b} \cdot \int_C (\mathbf{f} \wedge \mathbf{m}) \, d\ell = \mathbf{b} \cdot \int_S [\mathbf{n}(\nabla \cdot \mathbf{f}) - (\nabla \mathbf{f}) \cdot \mathbf{n}] \, dS \]

Since \( \mathbf{b} \) is arbitrary, we thus have

\[ \int_C (\mathbf{f} \wedge \mathbf{m}) \, d\ell = \int_S [\mathbf{n}(\nabla \cdot \mathbf{f}) - (\nabla \mathbf{f}) \cdot \mathbf{n}] \, dS \]

We now choose \( \mathbf{f} = \sigma \mathbf{n} \), and recall that \( \mathbf{n} \wedge \mathbf{m} = -\mathbf{s} \). One thus obtains

\[ -\int_C \sigma \mathbf{s} \, d\ell = \int_S [\mathbf{n}(\nabla \sigma \cdot \mathbf{n}) - \nabla(\sigma \mathbf{n}) \cdot \mathbf{n}] \, dS \]

\[ = \int_S [\mathbf{n}\nabla \sigma \cdot \mathbf{n} + \sigma \mathbf{n}(\nabla \cdot \mathbf{n}) - \nabla \sigma - \sigma (\nabla \mathbf{n}) \cdot \mathbf{n}] \, dS \]

We note that

\[ \nabla \sigma \cdot \mathbf{n} = 0 \text{ since } \nabla \sigma \text{ must be tangent to the surface } S, \]

\[ (\nabla \mathbf{n}) \cdot \mathbf{n} = \frac{1}{2} \nabla(\mathbf{n} \cdot \mathbf{n}) = \frac{1}{2} \nabla(1) = 0 , \]

and so obtain the desired result:

\[ \int_C \sigma \mathbf{s} \, d\ell = \int_S [\nabla \sigma - \sigma \mathbf{n} (\nabla \cdot \mathbf{n})] \, dS \]