Solutions for Homework #6

PROBLEM 1 (P.22 on page 375) A simple 1-dof system consists of a mass and a damper only, as shown in the figure below. The excitation is the real part of $F(t) = F_0 e^{i\omega t}$, where $F_0$ is real and positive.

\[ F(t) = F_0 e^{i\omega t}, \]

where: $m = 1$ kg, and $c = 10$ N sec / m.

- Draw a free body diagram of the mass showing all the forces.
- Find the equation of motion.
- Find the frequency response function $H_x(\omega)$, which is defined as the ratio of the response $X e^{i\omega t}$ to the input $F_0 e^{i\omega t}$. Sketch the magnitude of $|H_x(\omega)|$.
- At very low frequency, damping is dominant. Find the response magnitude $|X|$ and the approximate phase angle between the force and the response when $\omega = 0.5$ rad /sec. Let $F_0 = 5.0$ N.
- At high frequency, the system is mass controlled. Find the response magnitude $|X|$ and the approximate phase angle between the force and the response when $\omega = 200.00$ rad /sec.

SOLUTION

For the single degree of freedom system, the forcing function is of the following form:

\[ F(t) = \text{Re} \left( F_0 \cdot e^{i\omega t} \right) = \text{Re} \left( F_0 \cos(\omega t) + i F_0 \sin(\omega t) \right) = F_0 \cos(\omega t) \]  \hspace{1cm} (1)

We will evaluate the response using complex notation, and the response will be the real part of the result, namely:

\[ u(t) = \text{Re} \left( F_0 H(\omega) e^{i\omega t} \right) = \text{Re} \left( F_0 A(\omega) e^{i(\omega t - \phi)} \right) = F_0 A(\omega) \cos(\omega t - \phi) \]  \hspace{1cm} (2)

- The free body diagram of the structure is simply as follows. Note also that $u(t) = x(t)$

\[ c \ddot{u} \quad \quad \text{m} \ddot{u} \quad \quad F(t) \]

- The equation of motion is then formulated as follows:

\[ m \ddot{u} + c \dot{u} = F(t) = F_0 e^{i\omega t} \]  \hspace{1cm} (3)
• For the response of the structure \( u(t) = \tilde{u}(\omega) e^{i\omega t} \), we substitute in equation \(3\) to evaluate the frequency response function. Therefore we have (we cancel out the common non-zero exponential term):
\[
-\omega^2 m \tilde{u}(\omega) + i \omega c \tilde{u}(\omega) = F_0
\]
\[
\tilde{u}(\omega) = \frac{F_0}{-\omega^2 m + i \omega c}
\]
\[
H(\omega) = \frac{\tilde{u}(\omega)}{F_0} = \frac{1}{-\omega^2 m + i \omega c}
\]
\[
|H(\omega)| = \left| \frac{1}{-\omega^2 m + i \omega c} \right| = \frac{1}{\sqrt{(-\omega^2 m)^2 + (\omega c)^2}}
\]
Therefore, the absolute value of the transfer function \( H(\omega) \) is a hyperbolic function of the driving frequency \( \omega \), where:
\[
|H(\omega)| = \begin{cases} \infty & \text{for} \quad \omega \rightarrow 0 \\ 0 & \text{for} \quad \omega \rightarrow \infty \end{cases}
\]

We plot schematically the absolute value of the transfer function vs. the driving frequency \( \omega \).

\[
\frac{F_0}{|F_0|} |H(\omega)| = \frac{1}{\sqrt{(-\omega^2 m)^2 + (\omega c)^2}}
\]

• At very low frequency, damping is dominant. Find the response magnitude \(|X|\) and the approximate phase angle between the force and the response when \( \omega = 0.5 \) rad/sec. Let \( F_0 = 5.0 \) N.

We use equation \(4\) to evaluate the system response due to evaluate the response of the system as follows:
\[
|\tilde{u}(\omega)| = F_0 |H(\omega)| = F_0 \frac{1}{\sqrt{(-\omega^2 m)^2 + (\omega c)^2}}
\]
\[
|\tilde{u}(0.5)| = 5 \frac{1}{\sqrt{(-0.5^2 \cdot 1)^2 + (0.510)^2}} \approx 1.00
\]
\[
\tan(\phi) = \frac{\omega c}{-\omega^2 m} = -\frac{0.510}{0.5^2 \cdot 1} = -20 \Rightarrow \phi = -87^o = -\frac{\pi}{2} \text{ rad}
\]
The system response is almost $90^\circ$ out of phase with the applied forcing function, therefore damping is the control factor of the response.

- At high frequency, the system is mass controlled. Find the response magnitude $|X|$ and the approximate phase angle between the force and the response when $\omega = 200.00$ rad/sec.

We use equation (4) to evaluate the system response due to evaluate the response of the system as follows:

$$|\tilde{u}(\omega)| = F_0 |H(\omega)| = F_0 \frac{1}{\sqrt{\left(-\omega^2 m\right)^2 + (\omega c)^2}}$$

$$|\tilde{u}(200)| = 5 \frac{1}{\sqrt{\left(-200^2 \cdot 1\right)^2 + (200 \cdot 10)^2}} = 1.2485 \times 10^{-4} \approx 0.00$$

$$\tan(\phi) = \frac{\omega c}{-\omega^2 m} = -\frac{200 \cdot 10}{200^2 \cdot 1} = -0.05 \Rightarrow \phi = -2.86^\circ \approx 0^\circ \text{ rad}$$

The system response is almost in phase with the applied forcing function, therefore inertia (mass) is the control factor of the response.
Problem 2 (P. 23 on page 375) If the force input to a single degree of freedom vibration system is \( F(t) = F e^{i\omega t} \) and the steady-state output is \( X e^{i\omega t} \), then the frequency response function is \( H_{df}(\omega) = X e^{i\omega t} / |F| e^{i\omega t} \), which can be expressed as a magnitude and phase:

\[
H_{df}(\omega) = |H_{df}(\omega)| e^{i\phi} \\
X(t) = |F| |H_{df}(\omega)| e^{i\phi}
\]

- The Euler formula says \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \). Show that \( i = e^{\pi/2} \) and \(-1 = e^{i\pi} \).
- Find \( H_{X|F}(\omega) \) and \( H_{X|F}(\omega) \) the frequency response functions when the response is the velocity or the acceleration.

Hint: \( H_{X|F}(\omega) = \frac{X(t)}{F e^{i\omega t}} = |H_{X|F}(\omega)| e^{-i\phi} \quad H_{X|F}(\omega) = \frac{X(t)}{F e^{i\omega t}} \)

Generalize your answer by explaining the effect of phase angle when you take the time derivative of the response.

Solution

The Euler formula says:

\[
e^\theta = \cos(\theta) + i\sin(\theta)
\]

We use equation [1] as follows:

\[
e^{\pi/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i = i
\]

\[
e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0 = -1
\]

The equation of motion for the simple harmonic oscillator subjected to harmonic force is:

\[
m\ddot{u} + c\dot{u} + ku = F_0 e^{i\omega t}
\]

Notice that \( u(t) = X(t) \) in this present solution.

- For the velocity being of the form:
  \( \dot{u} = \tilde{u} e^{i\omega t} \)

  Displacement:
  \[
  u(t) = \int \dot{u} \, dt = \int \tilde{u} e^{i\omega t} \, dt = \frac{\tilde{u}}{i\omega} e^{i\omega t}
  \]

  Acceleration:
  \[
  \ddot{u}(t) = \frac{d}{dt} \dot{u}(t) = i\omega \tilde{u} e^{i\omega t}
  \]


\[
m i\omega \tilde{u} + c \tilde{u} + k \frac{\tilde{u}}{i\omega} = F_0
\]

\[
\tilde{u} = \frac{F_0}{m i\omega + c + k \frac{1}{i\omega}} = \frac{F_0}{i\omega \left( 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right) + 2i\xi \frac{\omega}{\omega_n}}
\]
\[ H_{u/F(u)} = \frac{i\omega}{k \left( 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right) + 2i\xi \frac{\omega}{\omega_n}} = i\omega H_{u/F(u)} \]  
\[ \text{[5]} \]

- For the acceleration being of the form: \( \ddot{u} = \ddot{u} e^{i\omega t} \)

Velocity:
\[ \dot{u}(t) = \int \ddot{u} \, dt = \int \ddot{u} e^{i\omega t} \, dt = \frac{\ddot{u}}{i\omega} e^{i\omega t} \]  
\[ \text{[6]} \]

Displacement:
\[ u(t) = \int \dot{u} \, dt = \int \frac{\ddot{u}}{i\omega} e^{i\omega t} \, dt = -\frac{\ddot{u}}{\omega^2} e^{i\omega t} \]

We substitute {3} in {2} and cancel out the common non-zero exponential term.

\[ m\ddot{u} + \frac{c}{i\omega} \dddot{u} - \frac{k}{\omega^2} \ddot{u} = F_0 \]
\[ \dddot{u} = \frac{F_0}{m - \frac{k}{\omega^2} + \frac{c}{i\omega} - \frac{k}{\omega^2} \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right] + 2i\xi \frac{\omega}{\omega_n}} \]  
\[ \dddot{u} = \frac{F_0}{m - \frac{k}{\omega^2} + \frac{c}{i\omega} - \frac{k}{\omega^2} \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right] + 2i\xi \frac{\omega}{\omega_n}} \]

\[ H_{u/F(u)} = \frac{-\omega^2}{k \left( 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right) + 2i\xi \frac{\omega}{\omega_n}} = -\omega^2 H_{u/F(u)} \]  
\[ \text{[8]} \]

We have proved:

- \( H_{u/F(u)} = i\omega H_{u/F(u)} \)

The velocity is 90° out of phase with the displacement, i.e. when \( u = \max, \dot{u} = 0 \).

- \( H_{u/F(u)} = -\omega^2 H_{u/F(u)} \)

The acceleration is proportional to the displacement, but they have opposite direction i.e. when \( u = \max, \ddot{u} = \min \) or \( |\dddot{u}| = \max \).