Beam Bending - Review

-- Plane sections remain plane.

\[ \rho : \text{radius of curvature}; \quad \kappa = \frac{1}{\rho} : \text{curvature}; \quad M : \text{moment} \]

Compatibility:  \[ \varepsilon_{xx} = -\frac{y}{\rho} \]

Independent of material behavior

Equilibrium:  \[ M = \int_{A} -\sigma_{xx} y \, dA \]

Independent of material behavior
Elastic beam bending

Elastic material behavior:

\[ \sigma_{xx} = E \varepsilon_{xx} \]

\[ \sigma_{xx} = E \varepsilon_{xx} = -E \left( \frac{y}{\rho} \right) \]

\[ M = \int_{A} -\sigma_{xx} y \, dA = \frac{E}{\rho} \int_{A} y^2 \, dA \]

\[ M = EI / \rho \]

\[ I = \int_{A} y^2 \, dA \]

For rectangular beams:

\[ I = BH^3 / 12 \]

For I-beams:

\[ I = (BH^3 - bh^3) / 12 \]

Linear stress distribution

Linear moment-curvature relationship
Elastic -plastic beam bending

elastic-perfectly plastic material behavior

nonlinear Moment-curvature relationship

In the elastic regime \( \sigma_{xx} = -\frac{M_y}{I} \rightarrow |\sigma_{max}| = M_{max}/I \)

Yielding initiates at the outer fibers of the beam when \( |\sigma_{max}| = \sigma_y \). This corresponds to a bending moment \( M_e \):

\[
M_e = \frac{(\sigma_y I)}{y_{max}}
\]

for rectangular beams: \( M_e = \frac{(\sigma_y BH^2)}{6} \)
Elastic-plastic beam bending

For $M > M_e$ the beam is in the elastic-plastic regime: the core of the beam (between $y = -c$ and $y = c$) is in the elastic regime, while the outer fibers are in the plastic regime ($\sigma = \sigma_y$).

For $|y| = c$, $\varepsilon = -\sigma_y/E \rightarrow -y/\rho = -c/\rho = -\sigma_y/E \rightarrow$ the extension of the elastic region is given by:

$$c = \rho \left( \frac{\sigma_y}{E} \right)$$

The moment-curvature relationship is then given by:

$$M = \int_{A} -\sigma_{xx} y \, dA = \frac{E}{\rho} \int_{c}^{c} y^2 \, dA + \int_{-\sigma_y}^{\sigma_y} y \, dA + \int_{c}^{y_{\text{max}}} -\sigma_y y \, dA$$

For a rectangular beam:

$$M = \sigma_y \left( 3H^2 - 4c^2 \right) B/12$$

For very large curvatures $c \rightarrow 0$ and the moment approaches the limit moment $M_p$, where the entire section is in the plastic regime

$$M_p = \int_{A} -\sigma_y y \, dA$$

for rectangular beams: $M_p = (\sigma_y BH^2) / 4 = 1.5 \, M_e$
Elastic -plastic beam bending. Unloading /springback

Material behavior: elastic unloading

$$\Delta \sigma = E \Delta \varepsilon$$

The beam unloads elastically

$$\Delta (1/\rho) = \Delta M / EI = -M_{\text{loaded}} / EI$$

$$\Delta (1/\rho)_{\text{unloaded}} = (1/\rho)_{\text{loaded}} - M_{\text{loaded}} / EI$$

Stress distribution upon unloading

- Stress distribution in the loaded configuration: $$\sigma^{\text{loaded}}$$
- Change in stress upon unloading: $$\Delta \sigma = (M_{\text{loaded}} y) / I$$
- Residual stress distribution: $$\sigma_{\text{residual}}$$